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Generalized Verma Modules, the Cartan–Helgason Theorem, and the Harish-Chandra Homomorphism

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1. INTRODUCTION

This paper is a natural sequel to [6(a)]. Here we complete the solution of [1, Problem 35, p. 336], in the intended algebraic spirit. (See [6(c)] for another kind of solution of the same problem.)

Specifically, let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be a Cartan decomposition of a real semisimple Lie algebra, $\mathbf{a}$ a Cartan subspace of $\mathbf{p}$, $\mathbf{g} = \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{n}$ a corresponding Iwasawa decomposition, $\mathbf{G}$ the universal enveloping algebra of $\mathbf{g}$, $\mathbf{G}^{\mathbf{k}}$ the centralizer of $\mathbf{k}$ in $\mathbf{G}$, $\mathbf{A}$ the universal enveloping algebra of $\mathbf{a}$, and $p: \mathbf{G}^{\mathbf{k}} \to \mathbf{A}$ the Harish-Chandra homomorphism, defined to be the projection to $\mathbf{A}$ with respect to the decomposition $\mathbf{G} = \mathbf{A} \oplus (\mathbf{kG} + \mathbf{Gn})$. The problem is to prove "purely algebraically" Harish-Chandra's theorem [2, Sect. 4] that the image of p equals the algebra $\mathbf{A}_W$ of translated (by half the sum of positive restricted roots) restricted Weyl group invariants in $\mathbf{A}$. The proof should be valid for semisimple symmetric Lie algebras with splitting Cartan subspaces, over arbitrary fields of characteristic zero (see [1, Sect. 1.13] and [6(a)]). In [6(a)], such a proof was given for the fact that $p(\mathbf{G}^{\mathbf{k}}) \subset \mathbf{A}_W$. (The subalgebra $\mathbf{n}$ will be denoted $\mathbf{u}$ in the body of this paper, since the symbol $\mathbf{n}$ has another natural use; see Section 2.)

Now the assertion that $p(\mathbf{G}^{\mathbf{k}})$ is all of $\mathbf{A}_W$ is equivalent to the assertion of Chevalley's polynomial restriction theorem [3(a), p. 430, Theorem 6.10]—that the restriction map from $\mathbf{p}$ to $\mathbf{a}$ takes the algebra of $\mathbf{k}$-invariant polynomial functions on $\mathbf{p}$ onto the algebra of polynomial functions on $\mathbf{a}$ invariant under the restricted Weyl group W. In the paper, we give two suitably general proofs of Chevalley's theorem; see Theorem 5.1. Actually, one of these proofs is nothing more than a modification of Harish-Chandra's classical proof presented in [3(a), pp. 433–434]; see the Appendix. In the body of this paper, we present a proof similar in spirit to the Kostant–Steinberg–Varadarajan proof of Chevalley's

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polynomial restriction theorem for Cartan subalgebras (cf. [1, Sect. 7.3] or [4, Sect. 23.1]). Our method, which is based on generalized Verma modules (see Sect. 2), will also be useful in subsequent work.

Specifically, we use such modules to give a natural Lie-algebraic proof of the Cartan–Helgason theorem (see [9, Theorem 3.3.1.1] and [3(c), Chap. III, Sect. 3]) characterizing the finite-dimensional irreducible **g**-modules which are spherical (i.e., which admit a nonzero **k**-invariant vector) (see Theorem 3.12 and Corollaries 3.13 and 3.14). The setting is that of a semisimple symmetric Lie algebra **g** with splitting Cartan subspace, over a field of characteristic zero "large enough" so that the relevant Cartan subalgebra of **g** splits. The idea that generalized Verma modules could be used to prove a generalized Cartan–Helgason theorem was originally suggested by the application of relative homological algebra to the generalized Bernstein–Gelfand–Gelfand resolution in [6(d)]. (The present paper is logically independent of [6(d)], however.)

Our argument requires a certain crucial limitation on the weight-space decomposition of the **k**-invariant vector in a spherical finite-dimensional irreducible **g**-module. This information is obtained straightforwardly in Section 4 by means of the finite group F of automorphisms of **g** defined in [6(b), Sect. 6]. (We do not need any hard results from [6(b)].) It is interesting to note that in the present paper, F plays a more flexible role in the theory of finite-dimensional **g**-modules than it plays in its original setting in [5, Chap. II, Sect. 6], where it enters into Kostant's analog of the Cartan–Helgason theorem. Specifically, in [5, Chap. II, Sect. 6], F is thought of as a subgroup of the algebraic torus in Aut **g** with Lie algebra **a**, and correspondingly, F acts trivially on any **a**-invariant in the finite-dimensional **g**-modules considered there. For us, the combinatorially defined F is not required to act with this restriction (see Section 4). To clarify this issue with an example, note that the three-dimensional irreducible module for the three-dimensional simple Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ (here $\mathfrak{k} = \mathfrak{so}(2, \mathbb{R})$) is spherical, and in our setup, F fixes the **k**-invariant vector. But in the setup of [5, Chap. II, Sect. 6], F does not fix the **k**-invariant vector in this module; instead, F fixes the **a**-invariant vector. In Section 4, we "force" F to fix the **k**-invariant vector in a spherical finite-dimensional irreducible **g**-module.

The use of F in Section 4 also gives another approach to part of Theorem 3.12; see the second Remark after Corollary 4.6.

In Section 5, we state and begin the proof of the polynomial restriction theorem, and in Section 6, we define and prove the W-invariance of the "spherical characters" associated with the spherical finite-dimensional irreducible **g**-modules. This W-invariance and the weight-space decomposition result of Section 4 are used in Section 7 to complete the proof of the restriction theorem. We also find that the **k**-invariant polynomial functions on **p** are spanned by certain concrete functions f_λ^μ associated with the spherical finite-dimensional irreducible **g**-modules (Theorem 7.5).

Finally, Section 8 is devoted to the surjectivity of the Harish–Chandra map and some standard consequences. For example, when the field is algebraically closed, we construct and index (by means of the W-orbits in $\mathbf{a}^*$) the set of equivalence classes of irreducible $\mathbf{g}$-modules admitting a nonzero $\mathbf{k}$-invariant vector. We also recover algebraically Harish–Chandra's result that the Harish–Chandra homomorphism is independent of the positive restricted root system used in its definition.

2. Generalized Verma Modules and Finite-Dimensional Irreducible Modules

We begin with some notation. Let k be a field of characteristic zero; $\mathbf{g}$ a split semisimple Lie algebra over k with splitting Cartan subalgebra $\mathbf{h}$; $\Delta \subset \mathbf{h}^*$ ($*$ denotes dual) the set of roots for $\mathbf{g}$ with respect to $\mathbf{h}$; $\Delta_+ \subset \Delta$ a positive system; $\Delta_- = -\Delta_+$; $l = \dim \mathbf{h}$; $\alpha_1, \ldots, \alpha_l \in \mathbf{h}^*$ the simple roots; e_i (resp., f_i) a nonzero element of the root space $\mathbf{g}^{\alpha_i}$ (resp., $\mathbf{g}^{-\alpha_i}$) for all $i \in 1, \ldots, l$, normalized so that $[e_i, f_i] = h_i$, where $\alpha_i(h_i) = 2$; $P \subset \mathbf{h}^*$ the set of dominant integral linear forms, i.e., $\{\lambda \in \mathbf{h}^* \mid \lambda(h_i) \in \mathbb{Z}_+ \text{ for all } i = 1, \ldots, l\}$ ($\mathbb{Z}_+$ denotes the set of nonnegative integers); $\rho = \frac{1}{2}\sum_{\varphi \in \Delta_+} \varphi \in \mathbf{h}^*$, so that $\rho(h_i) = 1$ for all $i = 1, \ldots, l$; and $r_1, \ldots, r_l \in \operatorname{Aut} \mathbf{h}^*$ the simple Weyl reflections (for all $\lambda \in \mathbf{h}^*$, $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$). Also, let S be an arbitrary subset of $\{1, \ldots, l\}$; $\mathbf{g}_S$ the subalgebra of $\mathbf{g}$ generated by $\{h_i, e_i, f_i\}_{i \in S}$; $\mathbf{h}_S$ the span of $\{h_i\}_{i \in S}$; $\Delta^S = \Delta \cap \coprod_{i \in S} \mathbb{Z}\alpha_i$ ($\mathbb{Z}$ the set of integers); $\Delta_+^S = \Delta_+ \cap \Delta^S$; $\Delta_-^S = \Delta_- \cap \Delta^S$; $\Delta_+(S) = \Delta_+ - \Delta_+^S$; and $\Delta_-(S) = \Delta_- - \Delta_-^S$.

Define the following subalgebras of $\mathbf{g}$: $\mathbf{n} = \coprod_{\varphi \in \Delta_+} \mathbf{g}^\varphi$; $\mathbf{n}^- = \coprod_{\varphi \in \Delta_-} \mathbf{g}^\varphi$; $\mathbf{n}_S = \coprod_{\varphi \in \Delta_+^S} \mathbf{g}^\varphi$; $\mathbf{n}_S^- = \coprod_{\varphi \in \Delta_-^S} \mathbf{g}^\varphi$; $\mathbf{u} = \coprod_{\varphi \in \Delta_+(S)} \mathbf{g}^\varphi$; $\mathbf{u}^- = \coprod_{\varphi \in \Delta_-(S)} \mathbf{g}^\varphi$; $\mathbf{r} = \mathbf{g}_S + \mathbf{h}$; and $\mathbf{p}_S = \mathbf{r} \oplus \mathbf{u}$ (a subalgebra because $[\mathbf{r}, \mathbf{u}] \subset \mathbf{u}$). Then $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}$; $\mathbf{g}_S = \mathbf{n}_S^- \oplus \mathbf{h}_S \oplus \mathbf{n}_S$; $\mathbf{n} = \mathbf{n}_S \oplus \mathbf{u}$; $\mathbf{n}^- = \mathbf{n}_S^- \oplus \mathbf{u}^-$; $\mathbf{r} = \mathbf{n}_S^- \oplus \mathbf{h} \oplus \mathbf{n}_S$; and $\mathbf{g} = \mathbf{u}^- \oplus \mathbf{p}_S$. Also, $\mathbf{g}_S$ is a split semisimple Lie algebra with splitting Cartan subalgebra $\mathbf{h}_S$, and $\mathbf{r}$ is a reductive Lie algebra with commutator subalgebra $\mathbf{g}_S$ and center a subalgebra of $\mathbf{h}$. As S varies among the subsets of $\{1, \ldots, l\}$, $\mathbf{p}_S$ varies among the parabolic subalgebras of $\mathbf{g}$ containing the Borel subalgebra $\mathbf{h} \oplus \mathbf{n}$. The reductive part of $\mathbf{p}_S$ is $\mathbf{r}$ and the nilpotent part of $\mathbf{p}_S$ is $\mathbf{u}$.

Let V be an $\mathbf{h}$-module (for example, a $\mathbf{g}$-module regarded as an $\mathbf{h}$-module by restriction). The *weight space* $V_\lambda \subset V$ corresponding to $\lambda \in \mathbf{h}^*$ is defined to be $\{v \in V \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathbf{h}\}$. Call λ a *weight* of V if $V_\lambda \neq 0$, and call the nonzero elements of V_λ *weight vectors* with weight λ.

Suppose that V is a $\mathbf{g}$-module. A weight vector $v \in V$ is called a *highest weight vector* if $\mathbf{n} \cdot v = 0$, and V is called a *highest weight module* if it is generated by a highest weight vector. In this case, the generating highest weight vector is uniquely determined up to nonzero scalar multiple, its weight is called the *highest weight* of V, and its weight space is called the *highest weight space* of V. The highest weight space is one-dimensional, V is the direct sum of its weight spaces, which are all finite dimensional, and all the weights of V are of the form

$\lambda - \sum_{i=1}^{l} n_i\alpha_i$ ($n_i \in \mathbb{Z}_+$), where $\lambda \in \mathbf{h}^*$ is the highest weight. These facts follow easily from the decomposition $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}$.

Now P indexes the set of (equivalence classes of) finite-dimensional irreducible $\mathbf{g}$-modules in the usual way—via the highest weight. Let $P_S = \{\lambda \in \mathbf{h}^* \mid \lambda(h_i) \in \mathbb{Z}_+$ for all $i \in S\}$. Then it is clear that there is a natural bijection, which we denote $\lambda \mapsto M(\lambda)$, between P_S and the set of (equivalence classes of) finite-dimensional irreducible $\mathbf{r}$-modules which are irreducible as $\mathbf{g}_S$-modules; the highest weight space relative to $\mathbf{h}_S$ and $\mathbf{n}_S$ of the $\mathbf{g}_S$-module $M(\lambda)$ is $\mathbf{h}$-stable, and λ is the corresponding weight.

Definitions. For all $\lambda \in \mathbf{h}^*$, let V^λ denote the corresponding *Verma module* for $\mathbf{g}$ (cf. [1, Sect. 7.1]): the $\mathbf{g}$-module $\mathrm{ind}(k_\lambda, \mathbf{g})$ induced (see [1, Sect. 5.1]) by the one-dimensional $\mathbf{h} \oplus \mathbf{n}$-module k_λ which is an $\mathbf{n}$-annihilated weight space (for $\mathbf{h}$) with weight λ. Also, for $\lambda \in P_S$, denote by $V^{M(\lambda)}$ the corresponding *generalized Verma module*: the $\mathbf{g}$-module $\mathrm{ind}(M(\lambda), \mathbf{g})$ induced by the $\mathbf{p}_S$-module $M(\lambda)$, viewed as an $\mathbf{r}$-module in the natural way and as a trivial $\mathbf{u}$-module.

Note that the Verma modules are precisely the generalized Verma modules for the case $S = \varnothing$ (i.e., $\mathbf{p}_S = \mathbf{h} \oplus \mathbf{n}$). Writing $\mathbf{G}$, $\mathbf{P}_S$, and $\mathbf{U}^-$ for the universal enveloping algebras of $\mathbf{g}$, $\mathbf{p}_S$, and $\mathbf{u}^-$, resp., we have $V^{M(\lambda)} = \mathbf{G} \otimes_{\mathbf{P}_S} M(\lambda)$ by definition, and the Poincare–Birkhoff–Witt theorem gives a natural linear isomorphism $V^{M(\lambda)} \simeq \mathbf{U}^- \otimes_k M(\lambda)$, where $\mathbf{U}^-$ acts by left multiplication on the first factor, and $\mathbf{r}$ acts by tensor product action. $V^{M(\lambda)}$ is a highest weight module with highest weight λ, and the highest weight space of $V^{M(\lambda)}$ coincides, under the natural identification of $M(\lambda)$ with the $\mathbf{p}_S$-submodule $1 \otimes M(\lambda)$ of $V^{M(\lambda)}$, with the highest weight space (relative to $\mathbf{h}_S$ and $\mathbf{n}_S$) of the $\mathbf{g}_S$-module $M(\lambda)$. The universal property of $V^{M(\lambda)}$ asserts that any $\mathbf{r}$-module map from $M(\lambda)$ into the $\mathbf{u}$-annihilated subspace of a $\mathbf{g}$-module extends uniquely to a $\mathbf{g}$-module map from $V^{M(\lambda)}$ into the $\mathbf{g}$-module. Of course, all these comments apply to the special case of Verma modules. For example, for $\lambda \in \mathbf{h}^*$, the Verma module V^λ is the universal highest weight module with highest weight λ, in the obvious sense.

Let $\lambda \in P_S$. Since $V^{M(\lambda)}$ is a highest weight module with highest weight λ, there is a surjection η: $V^\lambda \to V^{M(\lambda)}$ taking a highest weight vector, say v_0, generating V^λ to a highest weight vector generating $V^{M(\lambda)}$. Similarly, if $\mu \in P$ and R is the finite-dimensional irreducible $\mathbf{g}$-module with highest weight μ, then there is a surjection π: $V^\mu \to R$ taking a highest weight vector, say v_1, generating V^μ to a highest weight vector of R. The kernel of π is well known, by a classical theorem of Harish-Chandra (cf. [1, Sect. 7.2] or [4, Sect. 21.4]): It is the $\mathbf{g}$-submodule of V^μ generated by the highest weight vectors $f_i^{\mu(h_i)+1} \cdot v_1$ ($1 \leqslant i \leqslant l$). Regard $M(\lambda)$ in the natural way (see above) as an $\mathbf{r}$-submodule of $V^{M(\lambda)}$. Then the same theorem of Harish-Chandra implies that if X is the $\mathbf{r}$-submodule of V^λ generated by v_0, then the kernel of the surjection $\eta \mid X : X \to M(\lambda)$ is the $\mathbf{r}$-submodule of X generated by the highest weight vectors $f_i^{\lambda(h_i)+1} \cdot v_0$ for $i \in S$.

Proposition 2.1. *Let $\lambda \in P_S$, and let v_0 be a highest weight vector generating V^λ. Then there is a* $\mathbf{g}$*-module exact sequence*

$$\coprod_{i \in S} V^{r_i(\lambda+\rho)-\rho} \xrightarrow{\xi} V^\lambda \xrightarrow{\eta} V^{M(\lambda)} \longrightarrow 0,$$

where η takes v_0 to a highest weight vector generating $V^{M(\lambda)}$, and for each $i \in S$, ξ takes a highest weight vector generating $V^{r_i(\lambda+\rho)-\rho}$ to a nonzero multiple of the highest weight vector $f_i^{\lambda(h_i)+1} \cdot v_0 \in V^\lambda$.

Proof. All that we have to show is that $\operatorname{Ker} \eta$ is the $\mathbf{g}$-submodule Y of V^λ generated by $f_i^{\lambda(h_i)+1} \cdot v_0$ for $i \in S$. Now Y is the $\mathbf{g}$-submodule of V^λ generated by $\operatorname{Ker}(\eta \mid X)$ (see the discussion above), so that $Y \subset \operatorname{Ker} \eta$. On the other hand, there are $\mathbf{r}$-module maps $M(\lambda) \rightarrow X/\operatorname{Ker}(\eta \mid X) \rightarrow V^\lambda/Y$ whose composition is nonzero, and so we have a natural $\mathbf{r}$-module injection $\iota \colon M(\lambda) \rightarrow V^\lambda/Y$. The image of this injection is clearly annihilated by $\mathbf{u}$, so that ι extends to a $\mathbf{g}$-module map $V^{M(\lambda)} \rightarrow V^\lambda/Y$. Since $V^{M(\lambda)} = V^\lambda/\operatorname{Ker} \eta$, we see that $\operatorname{Ker} \eta \subset Y$. Q.E.D.

A module for a Lie algebra $\mathbf{l}$ (over k) is called *finitely semisimple* if it is a direct sum of finite-dimensional irreducible $\mathbf{l}$-submodules. Since k has characteristic zero, the tensor product of two finitely semisimple $\mathbf{l}$-modules is finitely semisimple.

Since $\mathbf{r}$ is reductive in $\mathbf{g}$, it is thus clear that $\mathbf{G}$ is finitely semisimple under the adjoint action of $\mathbf{r}$. It follows also that if a $\mathbf{g}$-module is generated by a finitely semisimple $\mathbf{r}$-submodule, then the $\mathbf{g}$-module is finitely semisimple under $\mathbf{r}$. Hence we have:

Proposition 2.2. *For all $\lambda \in P_S$, $V^{M(\lambda)}$ is finitely semisimple under* $\mathbf{r}$.

The following result is clear:

Proposition 2.3. *Let V be a finitely semisimple* $\mathbf{r}$*-module, let $v \in V$ be a weight vector with weight $\lambda \in \mathbf{h}^*$, and suppose that $\mathbf{n}_S \cdot v = 0$. Then $\lambda \in P_S$, and the* $\mathbf{r}$*-submodule of V generated by v is isomorphic to the irreducible* $\mathbf{r}$*-module $M(\lambda)$.*

Here is the main observation in this section:

Proposition 2.4. *Let R be the finite-dimensional irreducible* $\mathbf{g}$*-module with highest weight $\mu \in P$, let w_0 be a highest weight vector generating $V^{M(\mu)}$, and let $S' = \{1,\ldots, l\} - S$.* Then *$r_i(\mu + \rho) - \rho \in P_S$ for all $i \in S'$, and there is an exact* $\mathbf{g}$*-module sequence*

$$\coprod_{i \in S'} V^{M(r_i(\mu+\rho)-\rho)} \xrightarrow{\sigma} V^{M(\mu)} \xrightarrow{\tau} R \longrightarrow 0,$$

where τ takes w_0 to a highest weight vector of R, and for each $i \in S'$, σ takes a highest weight vector generating $V^{M(r_i(\mu+\rho)-\rho)}$ to a nonzero multiple of the highest weight vector $f_i^{\mu(h_i)+1} \cdot w_0 \in V^{M(\mu)}$.

Proof. By Proposition 2.1 and the Harish–Chandra theorem cited before it, the map τ in the statement of the proposition has as its kernel the $\mathbf{g}$-submodule of $V^{M(\mu)}$ generated by the highest weight vectors $f_i^{\mu(h_i)+1} \cdot w_0$ as i ranges through S'. (These vectors are nonzero because $V^{M(\mu)} \simeq \mathbf{U}^- \otimes_k M(\mu)$ and $f_i \in \mathbf{u}^-$ for $i \in S'$.) Now the weight of $f_i^{\mu(h_i)+1} \cdot w_0$ ($i \in S'$) is $r_i(\mu + \rho) - \rho$. Hence Propositions 2.2 and 2.3 imply that for $i \in S'$, $r_i(\mu + \rho) - \rho \in P_S$, and the $\mathbf{r}$-submodule of $V^{M(\mu)}$ generated by $f_i^{\mu(h_i)+1} \cdot w_0$ is isomorphic to $M(r_i(\mu + \rho) - \rho)$. Thus the $\mathbf{g}$-submodule of $V^{M(\mu)}$ generated by this highest weight vector is an image of $V^{M(r_i(\mu+\rho)-\rho)}$. Q.E.D.

Notation. For an $\mathbf{r}$-module M, denote by Y^M the $\mathbf{g}$-module coinduced (see [1, Sect. 5.5]) by the $\mathbf{p}_S$-module which is M as an $\mathbf{r}$-module and which is trivial as a $\mathbf{u}$-module. Y^M is the $\mathbf{g}$-module $\mathrm{Hom}_{\mathbf{p}_S}(\mathbf{G}, M)$.

Now the contragredient of an induced module is naturally isomorphic to the module coinduced from the contragredient of the inducing module (see [1, Proposition 5.5.4]). Denoting contragredient module by *, we thus have:

PROPOSITION 2.5. *For all* $\lambda \in P_S$, $(V^{M(\lambda)})^*$ *is naturally isomorphic to* $Y^{M(\lambda)^*}$.

The following is clear from Propositions 2.4 and 2.5:

PROPOSITION 2.6. *In the notation of Proposition* 2.4, *the* $\mathbf{g}$-*submodule of* $Y^{M(\mu)^*}$ *which is the annihilator of* $\mathrm{Im}\,\sigma \subset V^{M(\mu)}$ *is naturally isomorphic to* R^*.

3. Spherical Finite-Dimensional Irreducible Modules

In this section, we shall use generalized Verma modules to give a natural algebraic proof and generalization (Theorem 3.12) of the Cartan–Helgason theorem (cf. [9, Theorem 3.3.1.1]). We shall work in the setting of semisimple symmetric Lie algebras; see [6(a)] for background material.

Let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the symmetric decomposition of a semisimple symmetric Lie algebra $(\mathbf{g}, \theta)$ over a field k of characteristic zero, and let $\mathbf{a}$ be a Cartan subspace of $\mathbf{p}$. Let $\mathbf{m}$ be the centralizer of $\mathbf{a}$ in $\mathbf{k}$, and $\mathbf{l}$ a Cartan subalgebra of $\mathbf{m}$; then $\mathbf{h} = \mathbf{l} \oplus \mathbf{a}$ is a Cartan subalgebra of $\mathbf{g}$.

Assume that $\mathbf{h}$ is a splitting Cartan subalgebra, so that in particular, $\mathbf{a}$ is a splitting Cartan subspace. Let $\Delta \subset \mathbf{h}^*$ be the set of roots of $\mathbf{g}$ with respect to $\mathbf{h}$, and $\Sigma \subset \mathbf{a}^*$ the set of restricted roots of $\mathbf{g}$ with respect to $\mathbf{a}$. Choose a positive system $\Sigma_+ \subset \Sigma$. Now $\mathbf{h}$ is a splitting Cartan subalgebra of the reductive Lie algebra $\mathbf{m} \oplus \mathbf{a}$. Fix a positive system Φ_+ of the set Φ of roots of $\mathbf{m} \oplus \mathbf{a}$ with respect to $\mathbf{h}$. Then there is a unique positive system $\Delta_+ \subset \Delta$ containing Φ_+ and whose set of nonzero restrictions to $\mathbf{a}$ is Σ_+. We have $\Phi = \{\varphi \in \Delta \mid \varphi \mid \mathbf{a} = 0\}$, $\Phi_+ = \{\varphi \in \Delta_+ \mid \varphi \mid \mathbf{a} = 0\}$, and $\Delta_+ = \Phi_+ \cup \{\varphi \in \Delta \mid \varphi \mid \mathbf{a} \in \Sigma_+\}$.

Let $l = \dim \mathbf{h}$, let $\alpha_1, \ldots, \alpha_l \in \Delta_+$ be the simple roots of $\mathbf{g}$ with respect to $\mathbf{h}$

associated with Δ_+, and choose $S \subset \{1, \ldots, l\}$ so that the α_i with $i \in S$ are those simple roots which vanish on $\mathbf{a}$. Then these α_i are the simple roots of $\mathbf{m} \oplus \mathbf{a}$ with respect to $\mathbf{h}$ associated with Φ_+.

We may apply Section 2 to the present context, taking for k, $\mathbf{g}$, $\mathbf{h}$, Δ_+, and S in Section 2 the correspondingly designated objects in Section 3. Using now the notation of Section 2, we thus have $\mathbf{r} = \mathbf{m} \oplus \mathbf{a}$ and $\mathbf{p}_S = \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{u}$. Also, $\mathbf{g} = \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{u}$ is the Iwasawa decomposition of $\mathbf{g}$ associated with θ, $\mathbf{a}$, and Σ_+.

The elements $h_i \in \mathbf{h}$ for $i \in S$ (see Section 2) lie in $\mathbf{l}$. Let $P_\mathbf{m} \subset \mathbf{l}^*$ denote the set of dominant integral linear forms of the reductive Lie algebra $\mathbf{m}$ with respect to $\mathbf{l}$, i.e., $\{\nu \in \mathbf{l}^* \mid \nu(h_i) \in \mathbb{Z}_+ \text{ for all } i \in S\}$. Then $P_\mathbf{m}$ indexes the set of (equivalence classes of) finite-dimensional irreducible $\mathbf{m}$-modules which are irreducible under $\mathbf{g}_S$ $(=[\mathbf{m}, \mathbf{m}])$; for all $\nu \in P_\mathbf{m}$, denote by $M_\mathbf{m}(\nu)$ the corresponding $\mathbf{m}$-module, so that ν is the highest weight of $M_\mathbf{m}(\nu)$, in the obvious sense. Note that if $\lambda \in P$ or P_S (see Section 2), then $\lambda \mid \mathbf{l} \in P_\mathbf{m}$. For all $\lambda \in P_S$, the $\mathbf{m}$-module obtained by viewing the ($\mathbf{g}_S$-irreducible) $\mathbf{r}$-module $M(\lambda)$ as an $\mathbf{m}$-module remains irreducible, and is clearly isomorphic to $M_\mathbf{m}(\lambda \mid \mathbf{l})$.

Recall that for an $\mathbf{r}$-module M, Y^M is the $\mathbf{g}$-module coinduced from the $\mathbf{p}_S$-module M viewed as the $\mathbf{r}$-module M and as a trivial $\mathbf{u}$-module. Since $\mathbf{p}_S \cap \mathbf{k} = \mathbf{m}$, [1, Proposition 5.5.8] yields:

Proposition 3.1. *For all $\lambda \in P_S$, the $\mathbf{g}$-module $Y^{M(\lambda)^*}$, when regarded as a $\mathbf{k}$-module by restriction, is naturally isomorphic to the $\mathbf{k}$-module coinduced from the irreducible $\mathbf{m}$-module $M_\mathbf{m}(\lambda \mid \mathbf{l})^*$ up to $\mathbf{k}$.*

In view of this proposition, the universal property of coinduced modules (see [1, Proposition 5.5.3]) implies that for a $\mathbf{k}$-module T and $\lambda \in P_S$,

$$\mathrm{Hom}_\mathbf{k}(T, Y^{M(\lambda)^*}) \simeq \mathrm{Hom}_\mathbf{m}(T, M_\mathbf{m}(\lambda \mid \mathbf{l})^*),$$

where on the right, T is viewed as an $\mathbf{m}$-module by restriction. Apply this to the trivial one-dimensional $\mathbf{k}$-module T, and note that the irreducible $\mathbf{m}$-module $M_\mathbf{m}(\lambda \mid \mathbf{l})^*$ is trivial if and only if $\lambda \mid \mathbf{l} = 0$.

Definitions. Denote by $X^\mathbf{k}$ the space of $\mathbf{k}$-invariants in a $\mathbf{k}$-module X (possibly a $\mathbf{g}$-module regarded as a $\mathbf{k}$-module by restriction). Call X *spherical* if $X^\mathbf{k} \neq 0$.

We conclude:

Proposition 3.2. *Let $\lambda \in P_S$. Then $Y^{M(\lambda)^*}$ is spherical if and only if $\lambda \mid \mathbf{l} = 0$. In this case, $M(\lambda)^* = M(-\lambda)$, and $\dim(Y^{M(-\lambda)})^\mathbf{k} = 1$.*

We shall determine all the finite-dimensional irreducible spherical $\mathbf{g}$-modules.

First note that since $\mathbf{k}$ is reductive in $\mathbf{g}$ [1, Proposition 1.13.3], we have $\dim R^\mathbf{k} = \dim(R^*)^\mathbf{k}$ for a finite-dimensional irreducible $\mathbf{g}$-module R. In particular, R is spherical if and only if R^* is spherical. We shall use these observations without further comment.

PROPOSITION 3.3. *Let R be the finite-dimensional irreducible $\mathfrak{g}$-module with highest weight $\mu \in P$. Then* $\dim R^{\mathfrak{k}} \leqslant 1$. *If $\mu \mid \mathfrak{l} \neq 0$, then R is not spherical. Suppose $\mu \mid \mathfrak{l} = 0$, and denote by y_0 a generator of the one-dimensional space $(Y^{M(-\mu)})^{\mathfrak{k}}$. Also, let w_0 be a highest weight vector generating $V^{M(\mu)}$, and denote by $\langle \cdot, \cdot \rangle$ the natural pairing between $Y^{M(-\mu)}$ and $V^{M(\mu)}$. Then R is spherical if and only if $\langle y_0, \operatorname{Im} \sigma \rangle = 0$ (in the notation of Proposition 2.4), or equivalently, if and only if $\langle y_0, f_i^{\mu(h_i)+1} \cdot w_0 \rangle = 0$ for all $i \in S' = \{1,..., l\} - S$.*

Proof. By Proposition 2.6, R^* is naturally ismorphic to the annihilator of $\operatorname{Im} \sigma$ in $Y^{M(\mu)*}$. Hence $(R^*)^{\mathfrak{k}}$ is isomorphic to the space of all $\mathfrak{k}$-invariants in $Y^{M(\mu)*}$ which annihilate $\operatorname{Im} \sigma$. Proposition 3.2 thus implies the first two assertions of the present proposition. Suppose then that $\mu \mid \mathfrak{l} = 0$. It is clear that R^* is spherical if and only if $\langle y_0, \operatorname{Im} \sigma \rangle = 0$. By Proposition 2.4, this is the case if and only if $\langle y_0, \mathbf{G} f_i^{\mu(h_i)+1} \cdot w_0 \rangle = 0$ for all $i \in S'$. But $\mathbf{G}$ is the product $\mathbf{KAU}$ of the universal enveloping algebras of $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{u}$. Since y_0 is $\mathfrak{k}$-invariant and $f_i^{\mu(h_i)+1} \cdot w_0$ is a highest weight vector, we see that R^* is spherical if and only if

$$\begin{aligned} 0 &= \langle y_0, \mathbf{KAU} f_i^{\mu(h_i)+1} \cdot w_0 \rangle \\ &= \langle \mathbf{K} \cdot y_0, \mathbf{AU} f_i^{\mu(h_i)+1} \cdot w_0 \rangle \\ &= k \langle y_0, f_i^{\mu(h_i)+1} \cdot w_0 \rangle \end{aligned}$$

for all $i \in S'$. Q.E.D.

Now by Proposition 2.4, $f_i^{\mu(h_i)+1} \cdot w_0$ $(i \in S')$ is contained in an irreducible $\mathfrak{r}$-module equivalent to $M(r_i(\mu + \rho) - \rho)$, and hence in an irreducible $\mathfrak{m}$-module equivalent to $M_{\mathfrak{m}}((r_i(\mu + \rho) - \rho) \mid \mathfrak{l})$. But $r_i(\mu + \rho) - \rho = \mu - (\mu(h_i) + 1)\alpha_i$, with $\mu(h_i) + 1 > 0$, and so the conditions $\mu \mid \mathfrak{l} = 0$ and $\alpha_i \mid \mathfrak{l} \neq 0$ imply that $(r_i(\mu + \rho) - \rho) \mid \mathfrak{l} \neq 0$. Hence in this case, $M_{\mathfrak{m}}((r_i(\mu + \rho) - \rho) \mid \mathfrak{l})$ is not the trivial $\mathfrak{m}$-module. Thus in the notation of the last proposition, if $u \mid \mathfrak{l} = 0$, $i \in S'$ and $\alpha_i \mid \mathfrak{l} \neq 0$, then $\langle y_0, f_i^{\mu(h_i)+1} \cdot w_0 \rangle = 0$. The criterion in the last proposition thus simplifies to:

PROPOSITION 3.4. *In the notation of Proposition 3.3, suppose that $\mu \mid \mathfrak{l} = 0$. Then R is spherical if and only if $\langle y_0, f_i^{\mu(h_i)+1} \cdot w_0 \rangle = 0$ for all $i \in S'$ such that $\alpha_i \mid \mathfrak{l} = 0$.*

Hence we have:

COROLLARY 3.5. *Suppose that no simple root α_i $(i \in \{1,..., l\})$ vanishes on $\mathfrak{l}$. Let R be the finite-dimensional irreducible module with highest weight $\mu \in P$. Then R is spherical if and only if $\mu \mid \mathfrak{l} = 0$, and in this case,* $\dim R^{\mathfrak{k}} = 1$.

We now want to remove the hypothesis in the first sentence of Corollary 3.5, among other things. We begin with some generalities.

Let B be the Killing form of $\mathbf{g}$. Then B is nonsingular on $\mathbf{a}$ [6(a), Lemma 2.2], so that B naturally induces a nonsingular symmetric bilinear form $(\cdot, \cdot)$ on $\mathbf{a}^*$ and an isometry from $\mathbf{a}^*$ to $\mathbf{a}$. The form $(\cdot, \cdot)$ is rational valued and positive definite on the rational span $\mathbf{a}_{\mathbb{Q}}^*$ of Σ, and Σ forms a (not necessarily reduced) system of roots in the real Euclidean space $\mathbf{a}_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} \mathbb{R}$; also, $\mathbf{a}^* = \mathbf{a}_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} k$ [6(a), Lemmas 2.1, 2.2, and 2.4].

For all $\varphi \in \Sigma$, let $x_\varphi \in \mathbf{a}$ be the image of φ under the isometry from $\mathbf{a}^*$ to $\mathbf{a}$, and let $h_\varphi = 2x_\varphi/(\varphi, \varphi) \in \mathbf{a}$, so that $\varphi(h_\varphi) = 2$.

Let $\Sigma' = \{\varphi \in \Sigma \mid 2\varphi \notin \Sigma\}$. Then Σ' is a reduced system of roots. Let $\Pi' \subset \Sigma' \cap \Sigma_+$ denote the corresponding set of simple roots, so that Π' is a basis of $\mathbf{a}^*$. Hence $\{h_\varphi \mid \varphi \in \Pi'\}$ is a basis of $\mathbf{a}$; let $\lambda_1, ..., \lambda_n$ ($n = \dim \mathbf{a}$) be the corresponding dual basis of $\mathbf{a}^*$. We note the following for later use:

LEMMA 3.6. *Let $P_\Sigma = \{\lambda \in \mathbf{a}^* \mid \lambda(h_\varphi) \in \mathbb{Z}_+$ for all $\varphi \in \Sigma_+\}$. Then P_Σ is the set of nonnegative integral linear combinations of $\lambda_1, ..., \lambda_n$.*

Now the decomposition $\mathbf{h} = \mathbf{l} \oplus \mathbf{a}$ enables us to identify $\mathbf{h}^*$ with $\mathbf{l}^* \oplus \mathbf{a}^*$; an element $\lambda \in \mathbf{a}^*$ is identified with the linear functional on $\mathbf{h}$ which agrees with λ on $\mathbf{a}$ and which vanishes on $\mathbf{l}$. The Killing form B provides in the natural way a nonsingular symmetric bilinear form $(\cdot, \cdot)$ on $\mathbf{h}^*$. This form extends our form $(\cdot, \cdot)$ on $\mathbf{a}^*$, and $(\mathbf{l}^*, \mathbf{a}^*) = 0$ [6(a), Sect. 2]. The set $P \subset \mathbf{h}^*$ of dominant integral linear forms is the set $\{\lambda \in h^* \mid 2(\lambda, \psi)/(\psi, \psi) \in \mathbb{Z}_+$ for all $\psi \in \Delta_+\}$, P_Σ (see Lemma 3.6) is $\{\lambda \in \mathbf{a}^* \mid 2(\lambda, \varphi)/(\varphi, \varphi) \in \mathbb{Z}_+$ for all $\varphi \in \Sigma_+\}$. We shall need:

LEMMA 3.7 (cf. [3(c), pp. 76–77]). *We have $2P_\Sigma \subset P$.*

Proof. By [6(a), Lemma 2.3] and the subsequent discussion, we may apply the results on "normal σ-systems" in [9, Sect. 1.1.3]. (In fact, we need only the most trivial of these results.) Let $\lambda \in 2P_\Sigma$ and let $\psi \in \Delta_+$. Denoting by $\eta \mapsto \bar{\eta}$ the natural projection from $\mathbf{h}^*$ to $\mathbf{a}^*$, we have $2(\lambda, \psi)/(\psi, \psi) = 2(\lambda, \bar{\psi})/(\psi, \psi)$. If $\bar{\psi} = 0$, then this expression is 0. Otherwise, $\bar{\psi} \in \Sigma_+$. If $\bar{\psi} = \psi$, then $(\psi, \psi) = (\bar{\psi}, \bar{\psi})$, and we get $2(\lambda, \psi)/(\psi, \psi) \in 2\mathbb{Z}_+$. If $\bar{\psi} \neq \psi$, then either $(\psi, \psi) = 2(\bar{\psi}, \bar{\psi})$, or else $(\psi, \psi) = 4(\bar{\psi}, \bar{\psi})$ and $2\bar{\psi} \in \Delta_+ \cap \Sigma_+$ (see the bottom of [9, p. 21]). In the first case, $2(\lambda, \psi)/(\psi\ \psi) = (\lambda, \bar{\psi})/(\bar{\psi}, \bar{\psi}) \in \mathbb{Z}_+$, and in the other case, $2(\lambda, \psi)/(\psi, \psi) = (\lambda, 2\bar{\psi})/(2\bar{\psi}, 2\bar{\psi}) \in \mathbb{Z}_+$. Hence $\lambda \in P$. Q.E.D.

Let B_θ be the nonsingular symmetric bilinear form on $\mathbf{g}$ defined by the condition $B_\theta(x, y) = -B(x, \theta y)$ for all $x, y \in \mathbf{g}$. For $\varphi \in \Sigma$, define the restricted root space $\mathbf{g}^\varphi$ to be $\{x \in \mathbf{g} \mid [a, x] = \varphi(a)x$ for all $a \in \mathbf{a}\}$; there should be no confusion with the similar notation for root spaces of $\mathbf{g}$ with respect to $\mathbf{h}$.

Fix $\varphi \in \Sigma_+$. Since B_θ is nonsingular on $\mathbf{g}^\varphi$ [6(a), Lemma 3.2], we may choose a B_θ-nonisotropic vector $e_\varphi \in \mathbf{g}^\varphi$. Now $\theta e_\varphi \in \mathbf{g}^{-\varphi}$, and by [6(a), Lemma 3.3],

$[e_\varphi, \theta e_\varphi]$ is a nonzero multiple of $h_\varphi \in \mathbf{a}$. Let f_φ be the nonzero multiple of θe_φ such that $[e_\varphi, f_\varphi] = h_\varphi$. We have $[h_\varphi, e_\varphi] = 2e_\varphi$ and $[h_\varphi, f_\varphi] = -2f_\varphi$. Let $\mathbf{g}_\varphi$ be the three-dimensional split semisimple Lie algebra spanned by h_φ, e_φ, and f_φ. Note that kh_φ is a splitting Cartan subalgebra of $\mathbf{g}_\varphi$.

Remark. Let $i \in S'$ such that $\alpha_i \mid \mathbf{l} = 0$ (cf. Proposition 3.4), and let $\mathbf{g}_i$ be the three-dimensional split semisimple Lie algebra spanned by h_i, e_i, and f_i (see the beginning of Section 2). Since $\alpha_i \mid \mathbf{l} = 0$, we see that $h_i \in \mathbf{a}$ and that θe_i is a nonzero multiple of f_i. Also, α_i is a restricted root as well as a root, and lies in Σ_+. Now e_i lies in the restricted root space $\mathbf{g}^{\alpha_i}$, and the fact that $[e_i, \theta e_i]$ is a nonzero multiple of h_i implies that $h_i = h_{\alpha_i}$ and that e_i is B_θ-nonisotropic (see [6(a), Lemma 3.3]). Hence in this case, h_i, e_i, f_i, and $\mathbf{g}_i$ may be taken as the h_φ, e_φ, f_φ, and $\mathbf{g}_\varphi$ of the last paragraph, for $\varphi = \alpha_i$.

Now $\mathbf{g}_\varphi$ is θ-stable. Let $\mathbf{k}_\varphi = \mathbf{k} \cap \mathbf{g}_\varphi$, $\mathbf{p}_\varphi = \mathbf{p} \cap \mathbf{g}_\varphi$, $\mathbf{a}_\varphi = \mathbf{a} \cap \mathbf{g}_\varphi$, $\mathbf{m}_\varphi = \mathbf{m} \cap \mathbf{g}_\varphi$, and $\mathbf{u}_\varphi = \mathbf{u} \cap \mathbf{g}_\varphi$. Then $(\mathbf{g}_\varphi, \theta \mid \mathbf{g}_\varphi)$ is a semisimple symmetric Lie algebra with symmetric decomposition $\mathbf{g}_\varphi = \mathbf{k}_\varphi \oplus \mathbf{p}_\varphi$; $\mathbf{k}_\varphi = k(e_\varphi + \theta e_\varphi)$; $\mathbf{p}_\varphi = kh_\varphi \oplus k(e_\varphi - \theta e_\varphi)$; $\mathbf{a}_\varphi = kh_\varphi$ is a splitting Cartan subspace of $\mathbf{p}_\varphi$; $\mathbf{m}_\varphi = 0$ and is the centralizer of $\mathbf{a}_\varphi$ in $\mathbf{k}_\varphi$; $\mathbf{u}_\varphi = ke_\varphi$; $\varphi \mid \mathbf{a}_\varphi$ is a restricted root of $\mathbf{g}_\varphi$ with respect to $\mathbf{a}_\varphi$ which forms a system of positive restricted roots; and $\mathbf{g}_\varphi = \mathbf{k}_\varphi \oplus \mathbf{a}_\varphi \oplus \mathbf{u}_\varphi$ is the associated Iwasawa decomposition.

DEFINITION. A $\mathbf{k}_\varphi$-module X is called *spherical* if $X^{\mathbf{k}_\varphi} \neq 0$, where the superscript denotes the space of invariants.

PROPOSITION 3.8. *Let R_m be the m-dimensional irreducible $\mathbf{g}_\varphi$-module (m a positive integer). Then R_m is spherical if and only if m is odd, or equivalently, if and only if the eigenvalues of the action of h_φ on R_m are even.*

Proof. Suppose that R_m is spherical. Then R_m contains a nonzero vector annihilated by $\mathbf{k}_\varphi$, which is a Cartan subalgebra of $\mathbf{g}_\varphi$. Extending the field if necessary, we get an m-dimensional irreducible module for a three-dimensional simple Lie algebra such that a *splitting* Caratn subalgebra admits a nonzero invariant vector. This implies that m is odd.

Conversely, suppose that m is odd, and let $\{v_{m-1}, v_{m-2}, \ldots, v_{-(m-1)}\}$ be a basis of R_m such that v_j is an eigenvector for the action of h_φ with eigenvalue $2j$ ($j = m-1, m-2, \ldots, -(m-1)$). Since $\mathbf{k}_\varphi$ is spanned by $e_\varphi + \theta e_\varphi$, and θe_φ is a nonzero multiple of f_φ, it is obvious that a nonzero $\mathbf{k}_\varphi$-invariant vector in R_m can be constructed by taking a linear combination of the v_j of the form $c_{m-1}v_{m-1} + c_{m-3}v_{m-3} + c_{m-5}v_{m-5} + \cdots + c_{-(m-1)}v_{-(m-1)}$ for suitable $c_j \in k$. Q.E.D.

The next two results, which relate $\mathbf{g}_\varphi$ to finite-dimensional irreducible $\mathbf{g}$-modules, are immediate.

PROPOSITION 3.9. *Let R be the finite-dimensional irreducible* $\mathbf{g}$*-module with* *highest weight* $\mu \in P$, *and let* $r_0 \in R$ *be a highest weight vector. Then the* $\mathbf{g}_\varphi$*-submodule* R_φ *of* R *generated by* r_0 *is the finite-dimensional irreducible* $\mathbf{g}_\varphi$*-module with highest* h_φ*-eigenvalue* $\mu(h_\varphi)$, *which must lie in* $\mathbb{Z}_+$. *In particular, the restriction to* $\mathbf{a}$ *of an element of* P *lies in* P_Σ (*see Lemma* 3.6).

PROPOSITION 3.10. *Let* $\mu \in P$, *and let* w_0 *be a highest weight vector generating* $V^{M(\mu)}$. *Then the* $\mathbf{g}_\varphi$*-submodule* V_φ *of* $V^{M(\mu)}$ *generated by* w_0 *is the Verma module for* $\mathbf{g}_\varphi$ *with highest* h_φ*-eigenvalue* $\mu(h_\varphi) \in \mathbb{Z}_+$. *The image of* V_φ *under the natural map* $\tau\colon V^{M(\mu)} \to R = V^{M(\mu)}/\mathrm{Im}\,\sigma$ (*see Proposition* 2.4) *is the* $\mathbf{g}_\varphi$*-module* R_φ *in Proposition* 3.9. *In particular,* $f_\varphi^{\mu(h_\varphi)+1} \cdot w_0$ *lies in* $\mathrm{Im}\,\sigma$ *and in fact generates the* $\mathbf{g}_\varphi$*-submodule* $V_\varphi \cap \mathrm{Im}\,\sigma$ *of* V_φ.

We now come to the point—the reason for introducing the subalgebras $\mathbf{g}_\varphi$.

PROPOSITION 3.11. *Let* R *be the finite-dimensional irreducible* $\mathbf{g}$*-module with highest weight* $\mu \in P$, *and suppose that* $\mu \mid \mathbf{l} = 0$. *Then* $\mu(h_\varphi) \in \mathbb{Z}_+$, *and the following conditions are equivalent*:

(i) $\mu(h_\varphi) \in 2\mathbb{Z}_+$;

(ii) *the* $\mathbf{g}_\varphi$*-module* R_φ *defined in Proposition* 3.9 *is spherical*;

(iii) $\langle y_0, f_\varphi^{\mu(h_\varphi)+1} \cdot w_0\rangle = 0$, *where* y_0, w_0 *and* $\langle \cdot, \cdot \rangle$ *are as in Proposition* 3.3.

Proof. The fact that $\mu(h_\varphi) \in \mathbb{Z}_+$ is contained in Proposition 3.9, and the equivalence of (i) and (ii) follows from Propositions 3.8 and 3.9.

Now $\langle y_0, w_0\rangle \neq 0$. In fact, $\langle y_0, w_0\rangle = 0$ would imply

$$\begin{aligned} 0 &= \langle \mathbf{K} \cdot y_0, \mathbf{AU} \cdot w_0\rangle = \langle y_0, \mathbf{KAU} \cdot w_0\rangle \\ &= \langle y_0, \mathbf{G} \cdot w_0\rangle = \langle y_0, V^{M(\mu)}\rangle, \end{aligned}$$

a contradiction. ($\mathbf{G}$, $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{U}$ are the obvious universal enveloping algebras.)

Let V_φ be the $\mathbf{g}_\varphi$-submodule of $V^{M(\mu)}$ defined in Proposition 3.10, and let $\zeta \in V_\varphi^*$ be the linear functional which takes $v \in V_\varphi$ to $\langle y_0, v\rangle$. The fact that $\langle y_0, w_0\rangle \neq 0$ implies that $\zeta \neq 0$. Also, ζ is $\mathbf{k}_\varphi$-invariant. By Propositions 2.5 and 3.2 applied to $\mathbf{g}_\varphi$ in place of $\mathbf{g}$, ζ is the *unique* (up to scalar multiple) $\mathbf{k}_\varphi$-invariant in V_φ^*. Let us apply Proposition 3.3 to $\mathbf{g}_\varphi$ in place of $\mathbf{g}$, and R_φ in place of R. Then ζ, f_φ, $\mu(h_\varphi)$, and w_0 play the roles of y_0, f_i, $\mu(h_i)$, and w_0, respectively. We conclude that R_φ is spherical if and only if $\zeta(f_\varphi^{\mu(h_\varphi)+1} \cdot w_0) = 0$, i.e., if and only if (iii) holds. Q.E.D.

The main result of this section is now easy:

THEOREM 3.12 (cf. [9, Theorem 3.3.1.1]). *Let* R *be the finite-dimensional irreducible* $\mathbf{g}$*-module with highest weight* $\mu \in P$. *Then* $\dim R^{\mathbf{k}} \leqslant 1$, *and the following conditions are equivalent*:

(i) *R is spherical* (*i.e.*, $\dim R^{\mathfrak{k}} = 1$);

(ii) $\mu \in 2P_\Sigma$, *i.e.*, μ *vanishes on the Cartan subalgebra* $\mathfrak{l}$ *of* $\mathfrak{m}$ *and* $\mu(h_\varphi) \in 2\mathbb{Z}_+$ *for all* $\varphi \in \Sigma_+$ (*cf. Lemma* 3.6);

(iii) μ *vanishes on* $\mathfrak{l}$ *and for all simple roots* α_i ($i \in \{1, ..., l\}$) *vanishing on* $\mathfrak{l}$, *we have* $\mu(h_i) \in 2\mathbb{Z}_+$.

Moreover, $2P_\Sigma \subset P$, *so that the set of highest weights of spherical finite-dimensional irreducible* $\mathfrak{g}$*-modules is precisely* $2P_\Sigma$. *This set is exactly the set of nonnegative even integral linear combinations of the basis elements* $\lambda_1, ..., \lambda_n$ *of* $\mathfrak{a}^*$ *defined before Lemma* 3.6.

Proof. Proposition 3.3 implies that $\dim R^{\mathfrak{k}} \leqslant 1$.

(i) $\Rightarrow$ (ii) First note that $\mu \mid \mathfrak{l} = 0$ (Proposition 3.3). Let $\varphi \in \Sigma_+$. Then by Proposition 3.10, $f_\varphi^{\mu(h_\varphi)+1} \cdot w_0 \in \operatorname{Im} \sigma$, and so $\langle y_0, f_\varphi^{\mu(h_\varphi)+1} \cdot w_0 \rangle = 0$ by Proposition 3.3. Thus $\mu(h_\varphi) \in 2\mathbb{Z}_+$ from Proposition 3.11 ((iii) $\Rightarrow$ (i)).

(ii) $\Rightarrow$ (iii) is clear from the Remark after Lemma 3.7.

(iii) $\Rightarrow$ (i) Let α_i be a simple root vanishing on $\mathfrak{l}$. By the Remark after Lemma 3.7, Proposition 3.11 is applicable to h_i and f_i in place of h_φ and f_φ. Thus (i) $\Rightarrow$ (iii) in that proposition implies that $\langle y_0, f_i^{\mu(h_i)+1} \cdot w_0 \rangle = 0$. Hence R is spherical by Proposition 3.4.

Thus the equivalence of the three conditions is proved, and the rest is a consequence of Lemmas 3.6 and 3.7. Q.E.D.

Two "extreme" cases are worth isolating (cf. the Remark below):

COROLLARY 3.13. *Suppose the Cartan subspace* $\mathfrak{a}$ *is a Cartan subalgebra of* $\mathfrak{g}$ (*i.e.*, $\mathfrak{m} = 0$). *Then the highest weights of the spherical finite-dimensional irreducible* $\mathfrak{g}$*-modules constitute the set* $2P$.

COROLLARY 3.14 (cf. Corollary 3.5). *Suppose that no simple root* α_i ($i \in \{1, ..., l\}$) *vanishes on* $\mathfrak{l}$. *Then the highest weights of the spherical finite-dimensional irreducible* $\mathfrak{g}$*-modules constitute the set* $\{\mu \in P \mid \mu$ *vanishes on* $\mathfrak{l}\}$, *and this is exactly the set* $2P_\Sigma$.

Remark. Corollary 3.13 is precisely the case in which *every* simple root $\alpha_1, ..., a_l$ vanishes on $\mathfrak{l}$.

4. THE FOURIER EXPANSION OF THE k-INVARIANT

Retain the notation and assumptions of Section 3. Let R be a spherical finite-dimensional irreducible $\mathfrak{g}$-module, $r_* \in R$ a nonzero $\mathfrak{k}$-invariant. The goal of this section is to obtain an important limitation on the "Fourier components" of the expansion of r_* in terms of the weight space decomposition of R. For this, we use the finite group F in the form in which it is introduced in [6(b), Sect. 6]. The only results from [6(b)] that we shall need are easy and are independent of the main results of [6(b)].

Let F be the group of Lie algebra automorphisms of $\mathbf{g}$ consisting of those automorphisms which act as the identity on $\mathbf{m} \oplus \mathbf{a}$ and as either $+1$ or -1 on the restricted root space $\mathbf{g}^{\varphi}$ for each $\varphi \in \Sigma$. Then F is abelian of order 2^n ($n = \dim \mathbf{a}$), and is in fact the product of n two-element groups [6(b), Proposition 6.1].

Notation. For an F-module V, let V^F denote the space of F-invariants (i.e., elements fixed by F) in V. For a $\mathbf{g}$-module U and $\lambda \in \mathbf{a}^*$, let $U_{(\lambda)}$ be the associated restricted weight space $\{u \in U \mid a \cdot u = \lambda(a)u \text{ for all } a \in \mathbf{a}\}$. (The purpose of the parentheses is to distinguish this from the notation for $\mathbf{h}$-weight spaces; see Section 2.) When we use this notation with $U = \mathbf{G}$, we refer to natural adjoint action of $\mathbf{g}$ on $\mathbf{G}$.

F acts naturally as a group of automorphisms of $\mathbf{G}$, by unique extension of its action on $\mathbf{g}$. Then $\mathbf{G}$ is the direct sum of its character spaces for F, and we have:

LEMMA 4.1. *Let L be the subgroup of $\mathbf{a}^*$ generated by 2Σ. Then $\mathbf{G}^F = \coprod_{\lambda \in L} \mathbf{G}_{(\lambda)}$, and the sum of the character spaces of $\mathbf{G}$ for nontrivial characters of F is $\coprod_{\lambda \in \mathbf{a}^* - L} \mathbf{G}_{(\lambda)}$.*

(See [6(b), Proposition 6.2] and its proof.)

DEFINITION. Let $\mathbf{g}_1$ be an F-invariant subalgebra of $\mathbf{g}$, and let $\mathbf{G}_1 \subset \mathbf{G}$ be the universal enveloping algebra of $\mathbf{g}_1$. *A* $(\mathbf{G}_1, F)$*-module* is a $\mathbf{G}_1$-module V which is also an F-module, such that $f \cdot (g \cdot v) = (f \cdot g) \cdot (f \cdot v)$ for all $f \in F$, $g \in \mathbf{G}_1$, and $v \in V$. (It is sufficient to check this condition for $g \in \mathbf{g}_1$.)

DEFINITION. In the same notation, define the algebra $\mathbf{G}_1[F]$ to be $\mathbf{G}_1 \otimes k[F]$ as a vector space (here $k[F]$ is the group algebra of F), with multiplication determined by the condition $(g \otimes f)(g' \otimes f') = g(f \cdot g') \otimes ff'$ for all $g, g' \in \mathbf{G}_1$ and $f, f' \in F$. This makes $\mathbf{G}_1[F]$ a well-defined associative algebra with identity $1 \otimes 1$ and with subalgebras $\mathbf{G}_1$ and $k[F]$. (In fact, $\mathbf{G}_1[F]$ is the smash product of the algebra $\mathbf{G}_1$ with the bialgebra $k[F]$, in the sense of [8, p. 155].)

Remark [6(b), Proposition 6.3]. The $(\mathbf{G}_1, F)$-modules may be naturally identified with the $\mathbf{G}_1[F]$-modules.

Fix $\lambda \in \mathbf{h}^*$ such that $\lambda \mid \mathbf{l} = 0$. Then $\lambda \in P_S$, and so we can form the $\mathbf{g}$-module $V^{M(\lambda)}$ induced from the one-dimensional $\mathbf{p}_S = \mathbf{m} \oplus \mathbf{a} \oplus \mathbf{u}$-module $M(\lambda)$ on which $\mathbf{m}$ and $\mathbf{u}$ act trivially, and on which $\mathbf{a}$ acts via the scalar multiplications determined by $\lambda \mid \mathbf{a}$. Recall that $V^{M(\lambda)} = \mathbf{G} \otimes_{\mathbf{P}_S} M(\lambda)$, $\mathbf{P}_S$ being the universal enveloping algebra of $\mathbf{p}_S$. Now F clearly preserves $\mathbf{p}_S$, and when F is made to act trivially on $M(\lambda)$, $M(\lambda)$ clearly becomes a $(\mathbf{P}_S, F)$-module. We may thus define the corresponding induced $(\mathbf{G}, F)$-module

$$V_F^{M(\lambda)} = \mathbf{G}[F] \otimes_{\mathbf{P}_S[F]} M(\lambda).$$

The second Remark after Proposition 6.6 in [6(b)] implies:

LEMMA 4.2. *There is a natural* $\mathfrak{g}$*-module isomorphism* (*unique up to nonzero scalar multiple*) *between* $V^{M(\lambda)}$ *and* $V_F^{M(\lambda)}$.

We shall thus identify these two modules.

Now $V^{M(\lambda)}$ is clearly the direct sum of its restricted weight spaces and is also the direct sum of its character spaces for F. Lemma 4.1 easily yields (cf. [6(b), Proposition 6.4]):

LEMMA 4.3. *Let* $L_+ \subset L \subset \mathbf{a}^*$ *be the set of nonnegative even integral linear combinations of the elements of* Σ_+. *Then*

$$(V^{M(\lambda)})^F = \coprod_{\nu \in L_+} (V^{M(\lambda)})_{(\lambda-\nu)} .$$

(*We are identifying* λ *with* $\lambda \mid \mathbf{a} \in \mathbf{a}^*$.) *Moreover, the sum of the character spaces of* $V^{M(\lambda)}$ *for nontrivial characters of* F *is* $\coprod_{\nu \in \mathbf{a}^* - L_+} (V^{M(\lambda)})_{(\lambda-\nu)}$.

Recall that the coinduced $\mathfrak{g}$-module $Y^{M(-\lambda)}$ is naturally isomorphic to the contragredient $\mathfrak{g}$-module $(V^{M(\lambda)})^*$ (Propositions 2.5 and 3.2). Now the dual of an F-module has a natural contragredient F-module structure, and it is easy to see that with the contragredient actions of $\mathfrak{g}$ and F, the dual of a $(\mathbf{G}, F)$-module is also a $(\mathbf{G}, F)$-module. In particular, $Y^{M(-\lambda)}$ is a $(\mathbf{G}, F)$-module in a natural way.

LEMMA 4.4. *F preserves* $\mathfrak{k}$.

Proof. For each $\varphi \in \Sigma$ and $f \in F$, f acts as the same scalar (either $+1$ or -1) on $\mathfrak{g}^\varphi$ as on $\theta\mathfrak{g}^\varphi = \mathfrak{g}^{-\varphi}$ (see the proof of [6(b), Proposition 6.1]. Since $\mathfrak{k}$ is spanned by $\mathfrak{m}$ and elements of the form $e + \theta e$ where $e \in \mathfrak{g}^\varphi$ for some $\varphi \in \Sigma$, we see that F preserves $\mathfrak{k}$. Q.E.D.

LEMMA 4.5. *Let* y_0 *be a genrator of the one-dimensional space* $(Y^{M(-\lambda)})^{\mathfrak{k}}$ (*see Proposition* 3.2). *Then* y_0 *is fixed by every element of* F.

Proof. Denoting by $\langle \cdot, \cdot \rangle$ the natural pairing between $Y^{M(-\lambda)}$ and $V^{M(\lambda)}$, we observe that y_0 is the unique (up to scalar multiple) nonzero element of $Y^{M(-\lambda)}$ such that $\langle y_0, \mathfrak{k} \cdot V^{M(\lambda)} \rangle = 0$. Let w_0 be a highest weight vector generating $V^{M(\lambda)}$. Then w_0 is fixed by every element of F, and $\langle y_0, w_0 \rangle \neq 0$, as in the proof of Proposition 3.11. Now for all $f \in F$,

$$\begin{aligned}\langle f \cdot y_0, \mathfrak{k} \cdot V^{M(\lambda)} \rangle &= \langle y_0, f^{-1} \cdot (\mathfrak{k} \cdot V^{M(\lambda)}) \rangle \\ &= \langle y_0, \mathfrak{k} \cdot V^{M(\lambda)} \rangle = 0,\end{aligned}$$

using Lemma 4.4. Thus $f \cdot y_0$ is a multiple of y_0. But

$$\langle f \cdot y_0 , w_0 \rangle = \langle y_0 , f^{-1} \cdot w_0 \rangle = \langle y_0 , w_0 \rangle,$$

and so $f \cdot y_0$ must equal y_0. Q.E.D.

COROLLARY 4.6. *Let $\lambda \in \mathbf{h}^*$ be such that $\lambda \mid \mathbf{1} = 0$, let y_0 be a generator of the one-dimensional space $(Y^{M(-\lambda)})^{\mathbf{k}}$ (see Proposition 3.2), denote by $\langle \cdot, \cdot \rangle$ the natural pairing between $Y^{M(-\lambda)}$ and $V^{M(\lambda)}$, and let $L_+ \subset \mathbf{a}^* \subset \mathbf{h}^*$ be the set of nonnegative even integral linear combinations of the elements of Σ_+. Then for every $\nu \in \mathbf{h}^* - L_+$, we have $\langle y_0 , (V^{M(\lambda)})_{\lambda-\nu} \rangle = 0$.* (Here the subscript indicates the weight space for $\mathbf{h}$, as in Section 2.)

Proof. Just combine Lemmas 4.3 and 4.5, and note that each restricted weight space of $V^{M(\lambda)}$ is the direct sum of its intersections with the weight spaces of $V^{M(\lambda)}$. Q.E.D.

Remark. Note that the group F does not enter into the statement of Corollary 4.6. We are now finished using F.

Remark. Corollary 4.6 provides an alternative proof of the implication (iii) $\Rightarrow$ (i) in Theorem 3.12: In the notation of Theorem 3.12(iii) and of Proposition 3.4, $\mu(h_i) \in 2\mathbb{Z}_+$ implies that $f_i^{\mu(h_i)+1} \cdot w_0 \in V^{M(\mu)}$ has weight $\mu - m\alpha_i$, where m is a positive odd integer. Since $m\alpha_i \notin L_+$, Corollary 4.6 and Proposition 3.4 give the result.

LEMMA 4.7. *Let $\mu \in 2P_\Sigma$, R the spherical finite-dimensional irreducible $\mathbf{g}$-module with highest weight μ (see Theorem 3.12), r^* a nonzero $\mathbf{k}$-invariant in the contragredient module R^*, and $\nu \in \mathbf{h}^* - L_+$ (see Corollary 4.6). Then r^* vanishes on the weight space $R_{\mu-\nu}$.*

Proof. Take $\lambda = \mu$ in Corollary 4.6, and recall that $R = V^{M(\mu)}/\mathrm{Im}\,\sigma$ and that $\langle y_0 , \mathrm{Im}\,\sigma \rangle = 0$ (Proposition 3.3). The induced action of y_0 on R agrees with the action of r^* on R (up to nonzero scalar multiple). Now just apply Corollary 4.6. Q.E.D.

LEMMA 4.8. *In the notation of Lemma 4.7, let r_0 be a highest weight vector of R and r_0' a lowest weight vector of R. Then $\langle r^*, r_0 \rangle \neq 0$ and $\langle r^*, r_0' \rangle \neq 0$.*

Proof. Just apply the usual argument, as in the proof of Proposition 3.11; for r_0' use the decomposition $\mathbf{G} = \mathbf{KAU}^-$ ($\mathbf{U}^-$ the universal enveloping algebra of $\mathbf{u}^-$) in place of the decomposition $\mathbf{G} = \mathbf{KAU}$. Q.E.D.

PROPOSITION 4.9. *Let $\mu \in 2P_\Sigma$, R the spherical finite-dimensional irreducible $\mathbf{g}$-module with highest weight μ, $\mu' \in \mathbf{h}^*$ the lowest weight of R, and $r_* \in R$ the unique (up to scalar multiple) nonzero $\mathbf{k}$-invariant (see Theorem 3.12). For all*

$\nu \in \mathbf{h}^*$, let $(r_*)_\nu$ be the component of r_* in R_ν with respect to the weight space decomposition of R. Then $(r_*)_\mu \neq 0$; $(r_*)_{\mu'} \neq 0$; and $(r_*)_\nu = 0$ unless $\nu - \mu' \in L_+$ (see Corollary 4.6). In particular, $\mu - \mu' \in L_+$, and $(r_*)_\nu = 0$ unless $\mu - \nu \in L_+$.

Proof. For all $\nu \in \mathbf{h}^*$, the weight space $(R^*)_\nu$ of R^* vanishes on the weight space R_λ if $\lambda \in \mathbf{h}^*$, $\lambda \neq -\nu$; and $(R^*)_\nu$ is nonsingularly paired to $R_{-\nu}$. Hence our result for R^* in place of R follows immediately from the last two lemmas. But R^* is a completely arbitrary spherical finite-dimensional irreducible $\mathbf{g}$-module. Q.E.D.

Remark. Note that Proposition 4.9 immediately implies the assertions of Lemmas 4.7 and 4.8.

5. The Polynomial Restriction Theorem

Our algebraic proof of Chevalley's polynomial restriction theorem will require futher information about spherical finite-dimensional irreducible $\mathbf{g}$-modules, and for this, we shall use the part of the restriction theorem already proved in [6(a)]. So it is appropriate to state the theorem now and to begin the proof, which will be completed in Section 7.

Let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be the symmetric decomposition of a semisimple symmetric Lie algebra over a field k of characteristic zero, $\mathbf{a} \subset \mathbf{p}$ a splitting Cartan subspace, and $W \subset \operatorname{Aut} \mathbf{a}^*$ the corresponding restricted Weyl group (see [6(a), Sect. 2]). W may be defined as follows: Since the Killing form of $\mathbf{g}$ is nonsingular on $\mathbf{a}$ [6(a), Lemma 2.2], we get a natural nonsingular symmetric bilinear form $(\cdot, \cdot)$ on $\mathbf{a}^*$ together with an isometry from $\mathbf{a}^*$ to $\mathbf{a}$. Let x_φ be the image in $\mathbf{a}$ of $\varphi \in \Sigma$, where $\Sigma \subset \mathbf{a}^*$ is the set of restricted roots. By [6(a), Lemma 2.2], $(\varphi, \varphi) \neq 0$. Define $h_\varphi = 2x_\varphi/(\varphi, \varphi) \in \mathbf{a}$, so that $\varphi(h_\varphi) = 2$. The Weyl reflection of $\mathbf{a}^*$ with respect to φ is the automorphism which takes $\lambda \in \mathbf{a}^*$ to $\lambda - \lambda(h_\varphi)\varphi$, and W is the group of isometries of $\mathbf{a}^*$ generated by these reflections.

For a finite-dimensional vector space V over k, denote by $S(V)$ the symmetric algebra of V. Since k is infinite, $S(V^*)$ is naturally isomorphic to the algebra of polynomial functions on V, the isomorphism taking the symmetric algebra product $f_1 f_2 \cdots f_r$ of elements f_i of V^* to the corresponding product $f_1 f_2 \cdots f_r$ of functions on V. We shall identify $S(V^*)$ with the algebra of polynomial functions on V.

Now $\mathbf{k}$ acts on $\mathbf{p}$, hence on $\mathbf{p}^*$ by contragredience, and thus on $S(\mathbf{p}^*)$ by unique extension by derivations. Denote by $S(\mathbf{p}^*)^{\mathbf{k}}$ the algebra of $\mathbf{k}$-invariants. The action of W on $\mathbf{a}^*$ by linear automorphisms extends uniquely to an action of W on $S(\mathbf{a}^*)$ by algebra automorphisms; let $S(\mathbf{a}^*)^W$ be the algebra of W-invariants.

Denote by $F: S(\mathbf{p}^*) \to S(\mathbf{a}^*)$ the restriction homomorphism, and let $F_* = F \mid S(\mathbf{p}^*)^{\mathbf{k}}$.

The restriction theorem states:

THEOREM 5.1 (cf. [3(a), p. 430, Theorem 6.10]). *F_* is an algebra isomorphism from $S(\mathbf{p}^*)^{\mathbf{k}}$ onto $S(\mathbf{a}^*)^W$.*

Now the injectivity of F_* and the fact that $F_*(S(\mathbf{p}^*)^{\mathbf{k}}) \subset S(\mathbf{a}^*)^W$ have been established in [6(a), Theorem 3.1]; what remains to be proved is the surjectivity.

Let $\mathbf{m}$ be the centralizer of $\mathbf{a}$ in $\mathbf{k}$, and let $\mathbf{l}$ be a Cartan subalgebra of $\mathbf{m}$; then $\mathbf{h} = \mathbf{l} \oplus \mathbf{a}$ is a Cartan subalgebra of $\mathbf{g}$. To prove the surjectivity, it is clearly sufficient to do so with k replaced by any field extension of itself. Replace k by an extension such that the corresponding extension of $\mathbf{h}$ is a splitting Cartan subalgebra of the extension of $\mathbf{g}$. Then all the hypotheses of Section 3 hold, and so we are free to use all of the results and notation in Sections 3 and 4.

DEFINITION. For all $\mu \in 2P_\Sigma$ (see Lemma 3.6) and $j \in \mathbb{Z}_+$, define $f_j^\mu \in S(\mathbf{p}^*)$ as follows. Let R be the spherical finite-dimensional irreducible $\mathbf{g}$-module with highest weight μ (see Theorem 3.12), let $r_* \in R$ be a nonzero $\mathbf{k}$-invariant, and let r^* be a nonzero $\mathbf{k}$-invariant in the (spherical finite-dimensional irreducible) $\mathbf{g}$-module R^*. Denoting by $(r_*)_\mu$ the μ-component of r_* with respect to the weight space decomposition of R, we have $\langle r^*, (r_*)_\mu\rangle \neq 0$ by Proposition 4.9, $\langle \cdot, \cdot \rangle$ being the natural pairing. Hence we may assume that $\langle r^*, (r_*)_\mu\rangle = 1$. Then f_j^μ is the homogeneous polynomial function of degree j on $\mathbf{p}$ such that for all $x \in \mathbf{p}$, $f_j^\mu(x) = \langle r^*, x^j \cdot r_*\rangle$, where x^j is in the universal enveloping algebra $\mathbf{G}$. Note that f_j^μ is independent of r^* and r_*, subject to the indicated normalization. Also, it is clear that f_j^μ is in fact a homogeneous polynomial function of degree j; to see this, expand x in terms of a basis of $\mathbf{p}$.

It will turn out that as μ and j vary, the f_j^μ span $S(\mathbf{p}^*)^{\mathbf{k}}$ (see Theorem 7.5).

LEMMA 5.2. *For all $\mu \in 2P_\Sigma$ and $j \in \mathbb{Z}_+$, $f_j^\mu \in S(\mathbf{p}^*)^{\mathbf{k}}$.*

Proof. Let T^j be the space of symmetric tensors in $\otimes^j (\mathbf{p}^*) = (\otimes^j \mathbf{p})^*$. Then T^j is a $\mathbf{k}$-module in a natural way, and the natural map ω from T^j into $S(\mathbf{p}^*)$ is a $\mathbf{k}$-module isomorphism onto the jth symmetric power $S^j(\mathbf{p}^*)$ of $\mathbf{p}^*$. Now $f_j^\mu \in S^j(\mathbf{p}^*)$, and it is sufficient to show that $g = \omega^{-1} f_j^\mu$ is a $\mathbf{k}$-invariant in T^j. But g is the element of $(\otimes^j \mathbf{p})^*$ such that

$$g(x_1 \otimes \cdots \otimes x_j) = (1/j!)\, \Sigma_\pi \langle r^*, x_{\pi(1)} \cdots x_{\pi(j)} \cdot r_*\rangle$$

for all $x_1, \ldots, x_j \in \mathbf{p}$; π ranges through the symmetric group of $\{1, \ldots, j\}$. Hence for all $y \in \mathbf{k}$,

$$\begin{aligned}(y \cdot g)(x_1 \otimes \cdots \otimes x_j) &= -(1/j!)\, \Sigma_\pi \sum_{i=1}^{j} \langle r^*, x_{\pi(1)} \cdots [y, x_{\pi(i)}] \cdots x_{\pi(j)} \cdot r_*\rangle \\ &= -(1/j!)\, \Sigma_\pi \langle r^*, (y x_{\pi(1)} \cdots x_{\pi(j)} - x_{\pi(1)} \cdots x_{\pi(j)} y) \cdot r_*\rangle \\ &= 0,\end{aligned}$$

since $y \cdot r^* = y \cdot r_* = 0$. Thus $y \cdot g = 0$. Q.E.D.

Since $F_*(S(\mathbf{p}^*)^{\mathbf{k}}) \subset S(\mathbf{a}^*)^W$, the last lemma implies:

LEMMA 5.3. *For all* $\mu \in 2P_\Sigma$ *and* $j \in \mathbb{Z}_+$, $F_*(f_j^\mu) \in S(\mathbf{a}^*)^W$.

This result will be used in the next section to obtain deeper information on spherical finite-dimensional irreducible modules.

6. Spherical Characters

Retain the notation and assumptions of Sections 3, 4 and the last part of Section 5.

Denote by $k[\mathbf{a}^*]$ the group algebra of the abelian group $\mathbf{a}^*$, and for all $\lambda \in \mathbf{a}^*$, let e^λ be the corresponding element of $k[\mathbf{a}^*]$. Then $e^{\lambda+\nu} = e^\lambda e^\mu$ for all $\lambda, \nu \in \mathbf{a}^*$, and $e^0 = 1$ in $k[\mathbf{a}^*]$. Moreover, the restricted Weyl group W (see Section 5) acts in a natural way as a group of algebra automorphisms of $k[\mathbf{a}^*]$, by unique linear extension of its action on $\mathbf{a}^*$. In particular, $we^\lambda = e^{w\lambda}$ for all $w \in W$ and $\lambda \in \mathbf{a}^*$.

DEFINITION. Let $\mu \in 2P_\Sigma$ and let R be the spherical finite-dimensional irreducible $\mathbf{g}$-module with highest weight μ (see Theorem 3.12). Let $r_* \in R$ and $r^* \in R^*$ be nonzero $\mathbf{k}$-invariants. For each $v \in \mathbf{h}^*$, define $(r_*)_\nu$ to be the ν-component of r_* with respect to the weight space decomposition of R. Normalize r_* and r^* so that $\langle r^*, (r_*)_\mu\rangle = 1$, as in the definition of f_j^μ in Section 5. Then the *spherical character* $\mathrm{ch}_S R$ of R is the element of $k[\mathbf{a}^*]$ defined by the formula

$$\mathrm{ch}_S R = \Sigma_{\nu\in\mathbf{a}^*} \langle r^*, (r_*)_\nu\rangle\, e^\nu.$$

Note that this sum is finite, and that $\mathrm{ch}_S R$ is independent of r^* and r_*, subject to the indicated normalization.

By Proposition 4.9, we have immediately:

LEMMA 6.1. *In the above notation,* $\mathrm{ch}_S R$ *has the form* $\Sigma_{\nu\in\mathbf{a}^*} c_\nu e^\nu$ $(c_\nu \in k)$, *where* $c_\nu = 0$ *unless* $\mu - \nu \in L_+$; $c_\mu = 1$; *and* $c_{\mu'} \neq 0$, *where* μ' *is the lowest weight of* R.

The next observation is obvious from the definitions:

LEMMA 6.2. *Write* $\mathrm{ch}_S R = \Sigma_{\nu\in\mathbf{a}^*} c_\nu e^\nu$ *as in the last lemma. Then for all* $a \in \mathbf{a}$ *and* $j \in \mathbb{Z}_+$, $f_j^\mu(a) = \Sigma_{\nu\in\mathbf{a}^*} c_\nu \nu(a)^j$ *(see Section 5).*

We next use Lemma 5.3 to establish the following basic fact:

LEMMA 6.3. *In the same notation,* $\mathrm{ch}_S R$ *is W-invariant.*

Proof. Write $\mathrm{ch}_S R = \Sigma_{\nu\in\mathbf{a}^*} c_\nu e^\nu$ $(c_\nu \in k)$, and let $w \in W$. Now Lemma 5.3

implies that for all $a \in \mathbf{a}$ and $j \in \mathbb{Z}_+$, $f_j^\mu(a) = f_j^\mu(wa)$, i.e., $\Sigma_{\nu \in \mathbf{a}^*}(c_\nu - c_{w\nu})\, \nu(a)^j = 0$. by Lemma 6.2. At this point, let us regard $S(\mathbf{a})$ as the algebra of polynomial functions on $\mathbf{a}^*$. Then $a^j \in S(\mathbf{a})$ is the function which takes $\nu \in \mathbf{a}^*$ to $\nu(a)^j$, and so the last equation becomes $\Sigma_{\nu \in \mathbf{a}^*}(c_\nu - c_{w\nu})\, a^j(\nu) = 0$. But as $a \in \mathbf{a}$ and $j \in \mathbb{Z}_+$ vary, the powers a^j span $S(\mathbf{a})$ (see for example [6(a), Lemma 3.5(ii)]). Hence $\Sigma_{\nu \in \mathbf{a}^*}(c_\nu - c_{w\nu})\, p(\nu) = 0$ for all polynomial functions p on $\mathbf{a}^*$. Since the point evaluations at a finite set of distinct points of $\mathbf{a}^*$ form a linearly independent set of linear functionals on $S(\mathbf{a})$, we conclude that $c_\nu = c_{w\nu}$ for all $\nu \in \mathbf{a}^*$. But then $w(\mathrm{ch}_{\mathrm{S}}\, R) = \mathrm{ch}_{\mathrm{S}}\, R$. Q.E.D.

We now set up a bit more notation to obtain a general fact about W-orbits. Let $\mathbf{h}_{\mathbb{Q}}^*$ be the rational span of the set Δ of roots in $\mathbf{h}$, and recall (Section 3) that $\mathbf{a}_{\mathbb{Q}}^*$ is the rational span of Σ in $\mathbf{a}^*$, which we regard in the usual canonical way as a subspace of $\mathbf{h}^*$. Then $h_{\mathbb{Q}}^* = \mathbf{a}_{\mathbb{Q}}^* \oplus (\mathbf{l}^* \cap \mathbf{h}_{\mathbb{Q}}^*)$, $\mathbf{h}^* = \mathbf{h}_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} k$, and $\mathbf{a}^* = \mathbf{a}_{\mathbb{Q}}^* \otimes_{\mathbb{Q}} k$ [6(a), Lemma 2.1]. Recall (Section 3) that $(\cdot, \cdot)$ is the natural form on $\mathbf{h}^*$ induced by the Killing form of $\mathbf{g}$. Let $D = \{\nu \in \mathbf{h}_{\mathbb{Q}}^* \mid (\nu, \varphi) \geqslant 0$ for all $\varphi \in \Delta_+\}$, the set of dominant linear forms (with respect to Δ_+) in $\mathbf{h}_{\mathbb{Q}}^*$; and similarly, let D_Σ be the set of dominant linear forms (with respect to Σ_+) in $\mathbf{a}_{\mathbb{Q}}^*$. Then $P \subset D$ and $P_\Sigma \subset D_\Sigma$. It is clear that $D_\Sigma \subset D$. Let W_0 be the Weyl group of $\mathbf{g}$ with respect to $\mathbf{h}$, so that W_0 may be regarded as a group of isometries of $\mathbf{h}_{\mathbb{Q}}^*$.

Now for all $w \in W$, there exists $w' \in W_0$ such that w' preserves $\mathbf{a}_{\mathbb{Q}}^*$ and agrees with w on $\mathbf{a}_{\mathbb{Q}}^*$ (see [9, Lemma 1.1.3.5 or Proposition 1.1.3.3]). Since the most general (closed) Weyl chamber in $\mathbf{a}_{\mathbb{Q}}^*$ (with respect to Σ) is of the form wD_Σ, which is contained in $w'D$, we see that every Weyl chamber in $\mathbf{a}_{\mathbb{Q}}^*$ is contained in some Weyl chamber in $\mathbf{h}_{\mathbb{Q}}^*$ (with respect to Δ). We have thus proved:

PROPOSITION 6.4 (cf. [3(b), p. 599]). *Two elements of $\mathbf{a}_{\mathbb{Q}}^*$ are conjugate under W_0 if and only if they are conjugate under W. That is, the W-orbits in $\mathbf{a}_{\mathbb{Q}}^*$ are precisely the nonempty intersections with $\mathbf{a}_{\mathbb{Q}}^*$ of the W_0-orbits in $\mathbf{h}_{\mathbb{Q}}^*$.*

An *extremal weight* of a finite-dimensional irreducible $\mathbf{g}$-module is a W_0-transform of its highest weight. By Lemmas 6.1 and 6.3, the last proposition gives:

LEMMA 6.5. *In the notation of Lemma 6.1, $c_\nu = 1$ for every extremal weight ν of R such that $\nu \in \mathbf{a}^*$. Such weights constitute the W-orbit of μ, and include μ'.*

Summarizing the above lemmas, we have:

THEOREM 6.6. *Let $\mu \in 2P_\Sigma$ and let R be the spherical finite-dimensional irreducible $\mathbf{g}$-module with highest weight μ. Then the spherical character $\mathrm{ch}_{\mathrm{S}}\, R$ of R is a W-invariant element of $k[\mathbf{a}^*]$. Furthermore, write $\mathrm{ch}_{\mathrm{S}}\, R = \Sigma_{\nu \in \mathbf{a}^*} c_\nu e^\nu$ $(c_\nu \in k)$. Then $c_\nu = 1$ for every extremal weight ν of R such that $\nu \in \mathbf{a}^*$, including μ and the* *lowest weight of* R, *and* $c_\nu = 0$ *unless* $\mu - \nu \in L_+$ *(see Corollary* 4.6). *The extremal weights of* R *lying in* $\mathbf{a}^*$ *constitute the* W*-orbit of* μ.

7. Proof of the Restriction Theorem

The rest of the proof of Theorem 5.1 (the surjectivity) is just an imitation of the Kostant–Steinberg–Varadarajan proof of Chevalley's polynomial restriction theorem for Cartan subalgebras (cf. [1, Section 7.3] or [4, Section 23.1]).

Definition. For $f \in S(\mathbf{a}^*)$ or $f \in k[\mathbf{a}^*]$, let $\operatorname{Sym} f$ denote the sum of the distinct W-conjugates of f.

Remark. It is clear that $\operatorname{Sym} f \in S(\mathbf{a}^*)^W$ for all $f \in S(\mathbf{a}^*)$, and $\operatorname{Sym} f$ is a W-invariant in $k[\mathbf{a}^*]$ for all $f \in k[\mathbf{a}^*]$.

Lemma 7.1. *As* μ *ranges through* $2P_\Sigma$ *and* j *ranges through* $\mathbb{Z}_+$, *the elements* $\operatorname{Sym} \mu^j$ *span* $S(\mathbf{a}^*)^W$.

Proof. By Lemma 3.6, $2P_\Sigma$ is a Zariski dense subset of $\mathbf{a}^*$. Hence the powers μ^j ($\mu \in 2P_\Sigma$, $j \in \mathbb{Z}_+$) span $S(\mathbf{a}^*)$ (see, for example, [6(a), Lemma 3.5(ii)]). The lemma is now obvious. Q.E.D.

Definitions. Recall that L (Lemma 4.1) is the subgroup of $\mathbf{a}^*$ generated by 2Σ, and that $L_+ \subset L$ (Corollary 4.6) is the set of nonnegative even integral linear combinations of elements of Σ_+. Let $I = \{\nu \in \mathbf{a}^* \mid \nu(h_\varphi) \in 2\mathbb{Z} \text{ for all } \varphi \in \Sigma\}$, so that $2P_\Sigma \subset I$ (cf. Lemma 3.6). Define a partial ordering on $\mathbf{a}^*$ by asserting that $\nu \leqslant \mu$ ($\mu, \nu \in \mathbf{a}^*$) if $\mu - \nu \in L_+$.

The elementary properties of roots and weights in abstract root systems imply:

Lemma 7.2. We have:

(1) $L \subset I$.

(2) *Every element of* I *is* W*-conjugate to an element of* $2P_\Sigma$.

(3) *For all* $\mu \in 2P_\Sigma$, $\{\nu \in 2P_\Sigma \mid \nu \leqslant \mu\}$ *is finite.*

We now have:

Lemma 7.3. *Let* $\mu \in 2P_\Sigma$ *and* $j \in \mathbb{Z}_+$, $j > 0$. *Then* $F_*(f_j^\mu)$ (*see* Section 5) *is of the form*

$$F_*(f_j^\mu) = \operatorname{Sym} \mu^j + \Sigma_\nu c_\nu \operatorname{Sym} \nu^j,$$

where ν *ranges through* $\{\nu \in 2P_\Sigma \mid \nu \leqslant \mu \text{ and } \nu \neq \mu\}$, *and* $c_\nu \in k$. *Also,* $F_*(f_0^\mu)$ *is a nonzero element of* k.

Proof. Let R be the spherical finite-dimensional irreducible module with highest weight μ. By Theorem 6.6, $\mathrm{ch}_S\, R = \Sigma_{\nu \in \mu - L_+} c_\nu e^\nu$, where $c_\mu = 1$, $c_\nu \in k$ and $c_{w\nu} = c_\nu$ for all $w \in W$. Thus parts (1) and (2) of the last lemma imply that

$$\mathrm{ch}_S\, R = \mathrm{Sym}\, e^\mu + \Sigma_\nu c_\nu\, \mathrm{Sym}\, e^\nu,$$

where ν is as in the statement of the present lemma. Now just apply Lemma 6.2. Q.E.D.

For all $\mu \in 2P_\Sigma$, let n_μ be the number of elements in $\{\nu \in 2P_\Sigma \mid \nu \leqslant \mu\}$ (see Lemma 7.2(3)). By induction on n_μ, Lemma 7.3 implies:

LEMMA 7.4. *Let $\mu \in 2P_\Sigma$ and $j \in \mathbb{Z}_+$. Then* $\mathrm{Sym}\, \mu^0 = 1$ *is a multiple of $F_*(f_0{}^\mu)$, and if $j > 0$,*

$$\mathrm{Sym}\, \mu^j = F_*(f_j{}^\mu) + \Sigma_\nu\, d_\nu F_*(f_j{}^\nu),$$

where ν ranges through $\{\nu \in 2P_\Sigma \mid \nu \leqslant \mu \text{ and } \nu \neq \mu\}$, and $d_\nu \in k$.

Lemmas 7.1 and 7.4 show that $S(\mathbf{a}^*)^W$ is spanned by elements of the form $F_*(f_j{}^\mu)$, where $\mu \in 2P_\Sigma$ and $j \in \mathbb{Z}_+$. Hence F_* is surjective, and the proof of Theorem 5.1 is complete. The injectivity in Theorem 5.1 also yields:

THEOREM 7.5. *As μ ranges through $2P_\Sigma$ and j ranges through $\mathbb{Z}_+$, the elements $f_j{}^\mu$ (see Section 5) span $S(\mathbf{p}^*)^{\mathbf{k}}$.*

8. SURJECTIVITY OF THE HARISH–CHANDRA HOMOMORPHISM AND SOME CONSEQUENCES

Let k, $\mathbf{g}$, $\mathbf{k}$, $\mathbf{p}$, $\mathbf{a}$, Σ, and W be as in the statement of Theorem 5.1. Choose a positive system $\Sigma_+ \subset \Sigma$ and let $\mathbf{u} = \coprod_{\varphi \in \Sigma_+} \mathbf{g}^\varphi$, so that $\mathbf{g}$ has Iwasawa decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{u}$ [6(a), Lemma 2.6]. Denoting by $\mathbf{G}$, $\mathbf{K}$, $\mathbf{A}$, and $\mathbf{U}$ the corresponding universal enveloping algebras, we have a natural linear isomorphism $\mathbf{G} \simeq \mathbf{K} \otimes \mathbf{A} \otimes \mathbf{U}$, giving rise to the decomposition $\mathbf{G} = \mathbf{A} \oplus (\mathbf{kG} + \mathbf{Gu})$ (see [6(a), Section 4]). Let $p\colon \mathbf{G} \to \mathbf{A}$ be the corresponding projection. Similarly, let $q\colon \mathbf{G} \to \mathbf{A}$ be the projection with respect to the decomposition $\mathbf{G} = \mathbf{A} \oplus (\mathbf{Gk} + \mathbf{uG})$.

Let $\delta = \frac{1}{2}\sum_{\varphi \in \Sigma_+} (\dim \mathbf{g}^\varphi)\, \varphi \in \mathbf{a}^*$, and let τ be the unique algebra automorphism of $\mathbf{A}$ which takes $a \in \mathbf{a}$ to $a - \delta(a)$. Now the group W of automorphisms of $\mathbf{a}^*$ acts on $\mathbf{a}$ by contragredience, and by unique extension, W acts on $\mathbf{A}$ as a group of algebra automorphisms. Let $\mathbf{A}^W$ be the algebra of W-invariants in $\mathbf{A}$.

Denote by $\mathbf{G}^{\mathbf{k}}$ the centralizer of $\mathbf{k}$ in $\mathbf{G}$. Let p_* be the map $(\tau \circ p) \mid \mathbf{G}^{\mathbf{k}}\colon \mathbf{G}^{\mathbf{k}} \to \mathbf{A}$, and let q_* be the map $(\tau^{-1} \circ q) \mid \mathbf{G}^{\mathbf{k}}\colon \mathbf{G}^{\mathbf{k}} \to \mathbf{A}$.

Let $g \mapsto g^t$ denote the transpose antiautomorphism of $\mathbf{G}$, i.e., the unique antiautomorphism which is -1 on $\mathbf{g}$. Then for all $x \in \mathbf{G}$, $q(x) = (p(x^t))^t$, and for all $a \in \mathbf{A}$, $\tau^{-1}(a) = (\tau(a^t))^t$. Hence we have the following result, which enables us to transfer properties of p_* and q_* back and forth:

PROPOSITION 8.1. *For all* $x \in \mathbf{G}^{\mathbf{k}}$, $q_*(x) = (p_*(x^t))^t$.

The main theorem states:

THEOREM 8.2 (cf. [2, p. 260, Theorem 1]). *The maps* p_* *and* q_* *are both algebra homomorphisms from* $\mathbf{G}^{\mathbf{k}}$ *onto* $\mathbf{A}^W$, *each with kernel* $\mathbf{G}^{\mathbf{k}} \cap \mathbf{G}\mathbf{k} = \mathbf{G}^{\mathbf{k}} \cap \mathbf{k}\mathbf{G}$.

Proof. The assertions about q_* follow from those about p_*, by Proposition 8.1. For everything but the surjectivity of p_*, see [6(a), Sect. 4]. The surjectivity of p_* follows easily (in the present generality) from Theorem 5.1 and the fact that $\operatorname{Im} p_* \subset \mathbf{A}^W$, just as in Harish–Chandra's original proof; the argument is a simple induction on degree and uses the symmetrization mapping (see [2, p. 263] or [3(a), p. 431]). Q.E.D.

Notation. The isomorphism from $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{G}\mathbf{k}$ onto $\mathbf{A}^W$ induced by p_* (resp., q_*) shall also be denoted p_* (resp., q_*).

COROLLARY 8.3. *The algebras* $S(\mathbf{p})^{\mathbf{k}}$ *(defined in the obvious way),* $S(\mathbf{p}^*)^{\mathbf{k}}$ *and* $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{G}\mathbf{k} = \mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{k}\mathbf{G}$ *are all isomorphic to the polynomial algebra on* $\dim \mathbf{a}$ *algebraically independent generators.*

Proof. Just combine Theorems 5.1 and 8.2 with Chevalley's theorem on finite groups generated by reflections (cf. [1, 11.1.14]). Q.E.D.

COROLLARY 8.4. *For all* $\lambda \in \mathbf{a}^*$, *define the homomorphism* p_λ (resp., q_λ) *from* $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{G}\mathbf{k}$ *to* k *to be* p_* (resp., q_*) *followed by evaluation at* λ *(identifying* $\mathbf{A}$ *with the algebra of polynomial functions on* $\mathbf{a}^*$*). Then:*

(1) *For* $\lambda, \mu \in \mathbf{a}^*$, $p_\lambda = p_\mu$ (resp., $q_\lambda = q_\mu$) *if and only if* $\lambda \in W\mu$.

(2) *If* k *is algebraically closed, then every homomorphism from* $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{G}\mathbf{k}$ *into* k *is of the form* p_λ (resp., q_λ) *for some* $\lambda \in \mathbf{a}^*$.

Proof. (1) Use the proof of [1, Proposition 7.4.7].

(2) Use the proof of [1, Proposition 7.4.8]. Q.E.D.

DEFINITIONS. Call a $\mathbf{g}$-module X *spherical* if $X^{\mathbf{k}} \neq 0$, and *strictly spherical* if $\dim X^{\mathbf{k}} = 1$, where the superscript denotes the space of $\mathbf{k}$-invariants. Define the *infinitesimal spherical function* of a strictly spherical module X to be the scalar-valued homomorphism on $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{G}\mathbf{k}$ induced by the action of this algebra on $X^{\mathbf{k}}$.

Since $\mathbf{k}$ is reductive in $\mathbf{g}$ [1, Proposition 1.13.3], we have by [7, Theorem 5.5] (cf. also [1, Théorème 9.1.12]):

PROPOSITION 8.5. *There is a natural bijection between the set of equivalence classes of strictly spherical irreducible* $\mathbf{g}$*-modules and the set of scalar-valued homomorphisms of* $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{Gk}$. *The correspondence associates to a strictly spherical irreducible module its infinitesimal spherical function.*

THEOREM 8.6. *If k is algebraically closed, then every irreducible* $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{Gk}$*-module is one dimensional. In particular, every spherical irreducible* $\mathbf{g}$*-module is strictly spherical, and Corollary* 8.4 *provides a natural bijection between the set of equivalence classes of spherical irreducible* $\mathbf{g}$*-modules and the set of W-orbits in* $\mathbf{a}^*$.

Proof. By [1, Proposition 1.7.10], the graded algebra associated with the natural filtration of the algebra $\mathbf{G}^{\mathbf{k}}$ is commutative and finitely generated. Hence by [1, Lemme 2.6.4] (Quillen's lemma), the commuting ring of any irreducible $\mathbf{G}^{\mathbf{k}}$-module is algebraic over k, and hence equals k, since k is algebraically closed. The commutativity of $\mathbf{G}^{\mathbf{k}}/\mathbf{G}^{\mathbf{k}} \cap \mathbf{Gk}$ (Theorem 8.2) now implies the first assertion of the theorem. [7, Theorem 5.5] (or [1, Théorème 9.1.12]) thus shows that every spherical irreducible $\mathbf{g}$-module is strictly spherical, and the rest of the theorem follows from Corollary 8.4 and Proposition 8.5. Q.E.D.

Assume now that the field k is "large enough" so that the hypothesis of Section 3 holds—that $\mathbf{h} = \mathbf{l} \oplus \mathbf{a}$ is a splitting Cartan subalgebra of $\mathbf{g}$. We shall now construct the strictly spherical irreducible $\mathbf{g}$-modules with infinitesimal spherical functions of the form q_λ ($\lambda \in \mathbf{a}^*$).

PROPOSITION 8.7. *Let $\lambda \in \mathbf{a}^*$, extend λ to $\mathbf{h}^*$ by making it vanish on $\mathbf{l}$, and let y_0 span the* $\mathbf{k}$*-invariant subspace of the strictly spherical* $\mathbf{g}$*-module $Y^{M(-\lambda)}$ (see Proposition* 3.2). *Then $\mathbf{G} \cdot y_0$ is finitely semisimple (see Section* 2) *as a* $\mathbf{k}$*-module. Let T be the sum of all the* $\mathbf{G}$*-submodules of $\mathbf{G} \cdot y_0$ not containing y_0, so that T does not contain y_0, and $Z_\lambda = G \cdot y_0/T$ is* $\mathbf{G}$*-irreducible. Then Z_λ is the unique (up to equivalence) strictly spherical irreducible* $\mathbf{g}$*-module with infinitesimal spherical function $q_{-(\lambda+\delta)}$. Moreover, $Z_\lambda \simeq Z_\mu$ ($\lambda, \mu \in \mathbf{a}^*$) if and only if $\lambda + \delta \in W(\mu + \delta)$. If $\mu \in 2P_\Sigma$, then Z_μ is equivalent to the spherical finite-dimensional irreducible* $\mathbf{g}$*-module with lowest weight $-\mu$ (cf. Theorem* 3.12), *and all the spherical finite-dimensional irreducible* $\mathbf{g}$*-modules arise in this way. Finally, if k is algebraically closed, the Z_λ ($\lambda \in \mathbf{a}^*$) exhaust the strictly spherical irreducible* $\mathbf{g}$*-modules, up to equivalence.*

Proof. Since $\mathbf{k}$ is reductive in $\mathbf{g}$, $\mathbf{G}$ is finitely semisimple under $\mathbf{k}$, and thus $\mathbf{G} \cdot y_0$ is also. The $\mathbf{G}$-irreducibility of Z_λ is clear. The infinitesimal spherical function of Z_λ is the same as that of $Y^{M(-\lambda)}$. To compute this, let w_0 be a highest weight vector generating $V^{M(\lambda)}$ and let $\langle\cdot,\cdot\rangle$ be the natural pairing between $Y^{M(-\lambda)}$ and $V^{M(\lambda)}$ (see Propositions 2.5 and 3.2). Then $\langle y_0, w_0\rangle \neq 0$, as in the proof of Proposition 3.11, and the fact that the infinitesimal spherical function of $Y^{M(-\lambda)}$ is $q_{-(\lambda+\delta)}$ is now straightforward from the definitions. Corollary 8.4(1) implies the condition for the equivalence of Z_λ and Z_μ, Corollary 8.4(2) gives the last assertion of the proposition, and the statements about the spherical finite-dimensional irreducible modules follow from Theorem 3.12 and Propositions 2.6 and 3.3. Q.E.D.

We can now prove the following result, obtained originally by Harish–Chandra in the context of Lie groups (see [2, p. 263, footnote 20]); we shall use the approach of [1, p. 232]:

THEOREM 8.8. *The maps p_* and q_* are independent of the choice of the positive system $\Sigma_+ \subset \Sigma$ used in their definition.*

Proof. The most general positive system in Σ is of the form $w\Sigma_+$ for some $w \in W$. Let δ' denote the analog of δ for $w\Sigma_+$. By [6(a), Lemma 2.5], we have $\delta' = w\delta$. Let $p_*{}^w$ and $q_*{}^w$ denote the maps p_* and q_* defined using the positive system $w\Sigma_+$ in place of Σ_+; for $\lambda \in \mathbf{a}^*$, define $p_\lambda{}^w$ and $q_\lambda{}^w$ correspondingly. By Proposition 8.1, it is sufficient to show that $q_* = q_*{}^w$, and by extending the field if necessary, we may assume for this purpose that the hypothesis of Proposition 8.7 holds. Let $\mu \in 2P_\Sigma$, and let R be the spherical finite-dimensional irreducible $\mathbf{g}$-module with lowest weight $-\mu$ (Theorem 3.12). Choose $w' \in W_0$ such that w' preserves $\mathbf{a}_{\mathbb{Q}}{}^*$ and agrees with w on $\mathbf{a}_{\mathbb{Q}}{}^*$ (see Section 6). Then with respect to the w'-transform of the given positive system of $\mathbf{g}$ with respect to $\mathbf{h}$, $-w\mu$ is the lowest weight of R. Proposition 8.7, applied to both Σ_+ and $w\Sigma_+$, now implies that R has infinitesimal spherical function equal to $q_{-(\mu+\delta)} = q_{-w(\mu+\delta)}$ (using the fact that $\operatorname{Im} q_* \subset \mathbf{A}^W$) and also equal to $q^w_{-(w\mu+\delta')} = q^w_{-w(\mu+\delta)}$. But by Lemma 3.6, $\{-w(\mu+\delta) \mid \mu \in 2P_\Sigma\}$ is a Zariski dense subset of $\mathbf{a}^*$. We conclude that $q_* = q_*{}^w$. Q.E.D.

APPENDIX

Here we shall briefly indicate how the classical proof [3(a), pp. 433–434] of the polynomial restriction theorem can be modified to give the theorem in the generality of Theorem 5.1 above.

In the notation of Theorem 5.1, we assume that F_* is known to map $S(\mathbf{p}^*)^k$ injectively into $S(\mathbf{a}^*)^W$. We want to prove the surjectivity, and we may assume that the field k is algebraically closed.

For all $x \in \mathbf{p}$, let $T_x = (\operatorname{ad} x)^2 \mid \mathbf{p}$. The characteristic polynomial of T_x is of the form

$$\det(\lambda - T_x) = \lambda^r + p_{r-1}(x)\,\lambda^{r-1} + \cdots + p_n(x)\lambda^n,$$

where λ is an indeterminate, $r = \dim \mathbf{p}$, $n = \dim \mathbf{a}$, and the p_j are polynomial functions on $\mathbf{p}$. Let K be the irreducible algebraic subgroup of the adjoint group of $\mathbf{g}$ corresponding to $\mathrm{ad}_{\mathbf{g}}\mathbf{k}$ (cf. [1, 1.13.13]). Then the p_j are clearly K-invariant polynomial functions on $\mathbf{p}$, and hence they lie in $S(\mathbf{p}^*)^{\mathbf{k}}$. Thus the proof of [3(a), p. 433, Lemma 6.19] shows that $S(\mathbf{a}^*)$ is integral over $\operatorname{Im} F_*$.

If $f, g \in S(\mathbf{p}^*)^{\mathbf{k}}$ and $f = gq$ where $q \in S(\mathbf{p}^*)$, then $q \in S(\mathbf{p}^*)^{\mathbf{k}}$; this follows easily from the semisimplicity of $S(\mathbf{p}^*)$ as a $\mathbf{k}$-module (cf. [3(a), p. 433, Lemma 6.17]). The fact that $S(\mathbf{p}^*)^{\mathbf{k}}$ is integrally closed now follows exactly as in the proof of [3(a) Lemma 6.18, p. 433]. Hence $\operatorname{Im} F_*$ is integrally closed.

Let $x \in \mathbf{a}_{\mathbb{Q}}^*$ (see Section 6) and $y \in \mathbf{a}^*$, and suppose that $f(x) = f(y)$ for all $f \in \operatorname{Im} F_*$. We claim that x and y are W-conjugate (cf. [3(a), p. 433, Lemma 6.20]). To see this, note first that $g(x) = g(y)$ for all $g \in S(\mathbf{g}^*)^{\mathbf{g}}$ (using the obvious notation). By Chevalley's polynomial restriction theorem for Cartan subalgebras [1, Théorème 7.3.5(i)], the restriction map to the Cartan subalgebra $\mathbf{h} = \mathbf{l} \oplus \mathbf{a}$ of $\mathbf{g}$ takes $S(\mathbf{g}^*)^{\mathbf{g}}$ onto the space of W_0-invariants in $S(\mathbf{h}^*)$, where W_0 is the Weyl group of $\mathbf{g}$ with respect to $\mathbf{h}$ (cf. Section 6). Hence x and y are W_0-conjugate (see the proof of [1, Proposition 7.4.7]). In particular, $y \in \mathbf{a}_{\mathbb{Q}}^*$, and so Proposition 6.4 implies that x and y are W-conjugate. This proves the claim.

The Galois theory argument on [3(a) p. 434] now shows that $S(\mathbf{a}^*)^W$ lies in the quotient field of $\operatorname{Im} F_*$. Since $\operatorname{Im} F_*$ is integrally closed and $S(\mathbf{a}^*)^W$ is integral over $\operatorname{Im} F_*$ by the above, we conclude that $\operatorname{Im} F_* = S(\mathbf{a}^*)^W$. This proves Theorem 5.1.

References

1. J. Dixmier, "Algèbres enveloppantes," Gauthier–Villars, Paris, 1974.
2. Harish-Chandra, Spherical functions on a semisimple Lie group. I, *Amer. J. Math.* **80** (1958), 241–310.
3. S. Helgason, (a) "Differential Geometry and Symmetric Spaces," Academic Press, New York, 1962.
   (b) Fundamental solutions of invariant differential operators on symmetric spaces, *Amer. J. Math.* **86** (1964), 565–601.
   (c) A duality for symmetric spaces with applications to group representations, *Advances in Math.* **5** (1970), 1–154.
4. J. Humphreys, "Introduction to Lie Algebras and Representation Theory," Springer–Verlag, New York, 1972.
5. B. Kostant, On the existence and irreducibility of certain series of representations, *in* "Lie Groups and Their Representations," Summer School of the Bolyai-János Math. Soc. (I. M. Gelfand, Ed.), pp. 231–329, Halsted Press, New York, 1975.
6. J. Lepowsky, (a) On the Harish-Chandra homomorphism, *Trans. Amer. Math. Soc.* **208** (1975), 193–218.
   (b) Existence of conical vectors in induced modules, *Ann. of Math.* **102** (1975), 17–40.
   (c) A generalization of H. Weyl's "unitary trick," *Trans. Amer. Math. Soc.* **216** (1976), 229–236.
   (d) Generalized Verma modules, loop space cohomology, and Macdonald-type identities, to appear.

7. J. Lepowsky and G. W. McCollum, On the determination of irreducible modules by restriction to a subalgebra, *Trans. Amer. Math. Soc.* **176** (1973), 45–57.
8. M. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.
9. G. Warner, "Harmonic Analysis on Semi-simple Lie Groups. I," Springer–Verlag, New York, 1972.